HIGHER SECONDARY
PLANE, SOLID AND CO-ORDINATE GEOMETRY

(INCLUDING MENSURATION)

=H. S. ELECTIVE MATHEMATICS, PAPER II=

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PREFACE

This book has been written in accordance with the syllabus of Higher Secondary Course of the Board of Secondary Education, West Bengal.

Regarding the subject-matter we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of typical examples have been worked out by way of illustrations and examples for exercise have been selected very carefully.

Important formulæ have been given for ready reference. A few questions of the recent years have been given at the end to give the students an idea of the standard of examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labour amply rewarded if the students find the book useful to them.

Corrections of misprints and suggestions for improvement will be thankfully received.

B. C. Das.

B. N. Mukherje

Higher Secondary
Elective Mathematics : Paper II

PLANE GEOMETRY

(Syllabus for Class IX)

To prove

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle, together with twice the rectangle contained by one of those sides and the projection of the other side upon it. [*Theorem 1 Pages 4-5*]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle, diminished by twice the rectangle contained by one of these sides and the projection of the other side upon it. [*Theorem 2 Pages 5-6*]

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally, and the converse. [*Theorem 3 Page 17*]

If two triangles are equiangular their corresponding sides are proportional, and the converse. [*Theorems 5 and 6 Pages 23-24*]

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar. [*Theorem 7 Page 25*]

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally. [*Theorem 8 and its converse Pages 35-36*]

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each of the perpendicular are similar to the whole triangle and to one another. [*Theorem 9 Page 37*]

The ratio of the areas of similar triangles is equal to the ratio of the squares on the corresponding sides. [*Theorem 10 Pages 38-39*]

(Chapters I—II, Pages 3—48)

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PLANE GEOMETRY

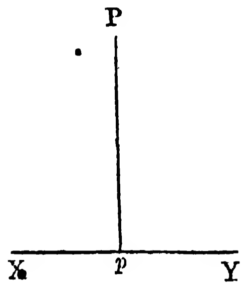
(Course for Class IX)

Chapter I

ORTHOGONAL PROJECTION

1·1. Orthogonal Projection.

The foot of the perpendicular drawn from a given external point upon a given straight line is called the **orthogonal projection** of the given point upon the given straight line.



In the adjoining figure, P is a point outside the straight line XY and Pp is the perpendicular drawn from the point P upon the straight line XY. Then p the foot of the perpendicular Pp on XY is the Orthogonal Projection of P on XY.

If from the extremities of a given line, straight or curved perpendiculars are drawn to a given straight line of indefinite length, the portion of the straight line intercepted between the feet of the perpendiculars is called the orthogonal projection of the given line upon the given straight line.

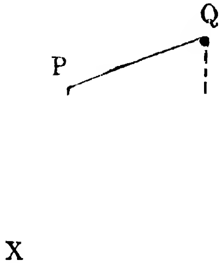


Fig. 1

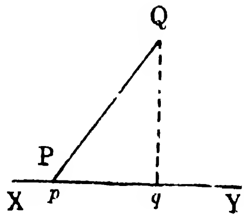


Fig. 2

In Figs. 1-3 the portion pq of the straight line XY intercepted between the feet of perpendiculars Pp and Qq drawn from the

extremities P and Q of the straight line PQ on XY is the Orthogonal Projection of the straight line PQ on XY. In Fig. 2, the point P is on the line XY. Hence, p, the foot of the

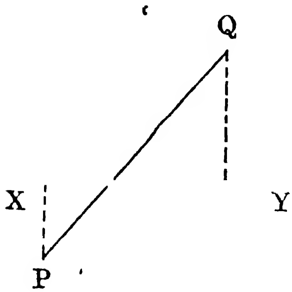


Fig. 3

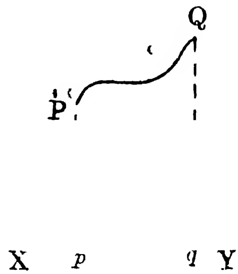
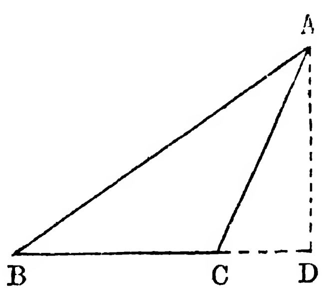


Fig. 4

perpendicular Pp, coincides with the point P of the straight line PQ. In Fig. 4, the straight line pq is the Orthogonal Projection of the curved line PQ on the straight line XY.

Theorem 1

In an obtuse-angled triangle the square on the side subtending **the obtuse angle** *is equal to the sum of the squares on the sides containing the obtuse angle* **together with** *twice the rectangle contained by one of those sides and the projection of the other upon it.*



Let ABC be a Δ in which the $\angle ACB$ is obtuse ; and from A let AD be drawn perpendicular to BC produced, so that CD is the projection of the side CA on BC.

It is reqd. to prove that $AB^2 = BC^2 + CA^2 + 2BC.CD$.

Proof. Because the $\angle D$ is a rt. angle

$\therefore\quad AB^2 = AD^2 + BD^2$. [Theorem of Pythagoras]

$= AD^2 + (CD + BC)^2$

$= AD^2 + CD^2 + BC^2 + 2BC.CD$

$= AC^2 + BC^2 + 2BC.CD$. [$\because$ $\angle D =$ a rt. angle]

Theorem 2

In any triangle the square on the side subtending an **acute angle** *is equal to the sum of the squares on the sides containing the acute angle* **diminished by** *twice the rectangle contained by one of those sides and the projection of the other side upon it.*

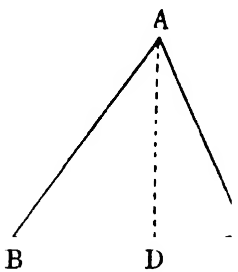


Fig. 1

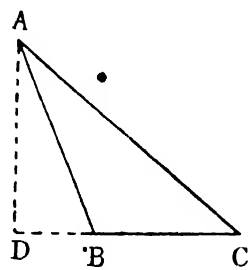


Fig. 2

Let ABC be a Δ in which the $\angle ACB$ is acute ; and from A let AD be drawn perp. to BC or BC produced, so that CD is the projection of CA on BC.

It is reqd. to prove that $AB^2 = AC^2 + BC^2 - 2BC.CD$.

Proof. Because the $\angle D$ is a rt. angle,

$\therefore\quad AB^2 = AD^2 + BD^2$ [Theorem of Pythagoras]

$= AD^2 + (BC - CD)^2$ [In Fig. 1, $BD = BC - CD$]

or, $\qquad = AD^2 + (CD - BC)^2 \qquad$ [In Fig. 2, $BD = CD - BC$]

$$= AD^2 + CD^2 + BC^2 - 2BC.CD$$

$$= AC^2 + BC^2 - 2BC.CD. \qquad [\because \angle D \text{ is a rt. angle}]$$

N. B. In the previous two theorems, it is noticed that in the $\triangle ABC$,

if (i) $AB^2 > AC^2 + BC^2$, the $\angle C$ will be obtuse,

and (ii) $AB^2 < AC^2 + BC^2$, the $\angle C$ will be acute.

It is needless to point out that if $AB^2 = AC^2 + BC^2$, then according to the Converse of the Theorem of Pythagoras, the $\angle C$ will be a rt. angle.

An Important Deduction : Apollonius' Theorem.

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

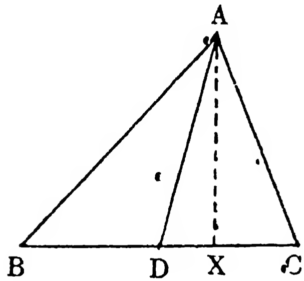


Fig. 1

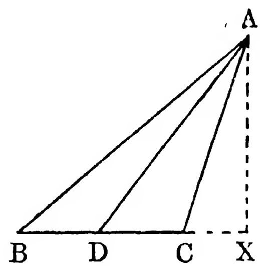


Fig. 2

In the $\triangle ABC$ let AD be the median bisecting the sides BC at the point D.

It is reqd. to prove that $AB^2 + AC^2 = 2(BD^2 + AD^2)$.

Construction. From the pt. A draw AX perp. to BC or BC produced.

Proof. Suppose AB and AC are unequal. Then of the two angles ADB and ADC one is obtuse and the other is acute. Let the $\angle ADB$ be obtuse, then the $\angle ADC$ will be acute.

$\therefore$ from the $\triangle ADB$, $AB^2 = BD^2 + AD^2 + 2BD.DX$.

[Theor. 1]

and from the $\triangle ADC$, $AC^2 = CD^2 + AD^2 - 2CD.DX$. [Theor. 2]

Adding these two results and remembering that $BD = CD$ ($\because$ D is the middle pt. of BC), we have

$$AB^2 + AC^2 = 2(BD^2 + AD^2).$$

N. B. If $AB = AC$, the proof of the theorem is easy enough.

12. Illustrative Examples.

Ex. 1. *The $\angle ACB$ of a $\triangle ABC$ is $60°$; prove that*

$$AB^2 = AC^2 + BC^2 - AC.BC.$$

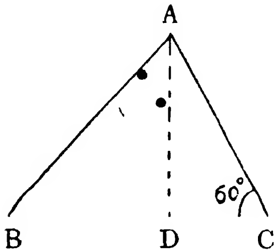


Fig. 1

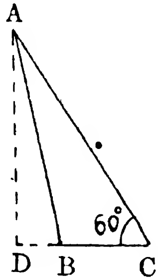


Fig. 2

Let ABC be a $\triangle$ in which the $\angle ACB = 60°$.

It is reqd. to prove that $AB^2 = AC^2 + BC^2 - AC.BC$.

Construction. From the pt. A draw AD perp. to BC or BC produced.

Proof. In the rt.-angled $\triangle ADC$, the $\angle ACD = 60°$ (Hyp.)

$\therefore$ $\angle DAC = 30°$.

$\therefore$ the side CD opposite to the $\angle DAC = \frac{1}{2}AC$. [In a rt.-angled $\triangle$ it can be easily proved that if one of its angles be $30°$, the side opposite to the $30°$ angle is equal to half the hypotenuse.]

Now, in the $\triangle ACB$, the $\angle ACB$ is acute.

$\therefore$ $AB^2 = AC^2 + BC^2 - 2BC.CD$. [Theor. 2]

$= AC^2 + BC^2 - AC.BC$. [$\because$ $CD = \frac{1}{2}AC$]

Ex. 2. *Prove that a triangle whose sides are 12 cms., 14 cms., and 18 cms. is an acute-angled triangle.*

Here the sides 18 cms. is the greatest side of the $\triangle$.

$\therefore$ the $\angle$ opposite to this side must be the greatest $\angle$ of the $\triangle$.

Now, if it can be proved that the $\angle$ opposite to this side is acute, then

the other two angles of the △ being less than this angle must also be acute and the △ will be an acute-angled one.

From Theorem 2, we know that in any triangle the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle.

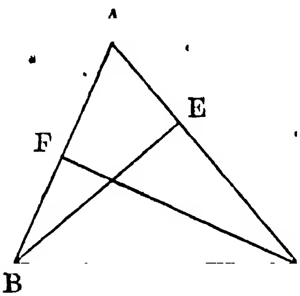
Now $18^2 < 12^2 + 14^2$.

∴ the angle opposite to the 18 cms. side is an acute angle.

∴ the other two angles of the △ being less than this angle must also be acute.

∴ all the three angles of the △ being acute, it must be an acute-angled one.

Ex. 3. *The angles B and C of a triangle ABC are acute and BE, CF are perpendiculars to AC, AB respectively; prove that* $BC^2 = AB.BF + AC.CE$.



Let ABC be a △ in which the ∠s B, C are acute and BE, CF are drawn perps. to AC, AB respectively.

It is reqd. to prove that

$$BC^2 = AB.BF + AC.CE.$$

Proof. Since the $\angle C$ is acute,

$$AB^2 = AC^2 + BC^2 - 2AC.CE.$$

$$\therefore \quad BC^2 = AB^2 - AC^2 + 2AC.CE.$$

In a similar way, since the $\angle B$ is acute

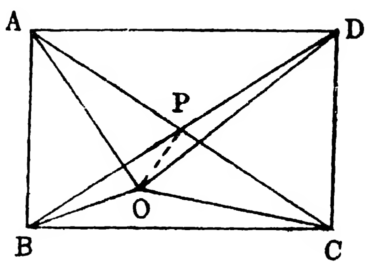
$$BC^2 = AC^2 - AB^2 + 2AB.BF.$$

Adding these two results, $2BC^2 = 2AC.CE + 2AB.BF$,

that is, $BC^2 = AC.CE + AB.BF$.

Ex. 4. *If O is any point within a rectangle ABCD, show that*

$$OA^2 + OC^2 = OB^2 + OD^2.$$



Let O be any point within the rect. $ABCD$. Join OA, OB, OC, OD.

It is reqd. to prove that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

Construction. Join AC, BD and let P be the pt. of their intersection. Join OP.

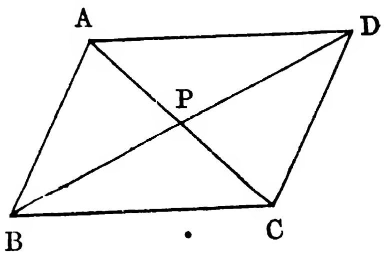
Proof. The diagonals AC, BD of the rect. $ABCD$ bisect each other at P.

$\therefore$ in the $\triangle AOC$, $OA^2+OC^2=2(AP^2+OP^2)$
and in the $\triangle BOD$, $OB^2+OD^2=2(BP^2+OP^2)$ } [Apollonius' Theorem]

But $AP=BP$ [$\because$ AC, BD being the diagonals of a rect. are equal]

$$\therefore\ OA^2+OC^2=OB^2+OD^2.$$

Ex. 5. *Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.*



Let the diagonals AC, BD of the par^m. $ABCD$ intersect at P.

It is reqd. to prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$.

Proof. Since the diagonals of a par^m. bisect each other,

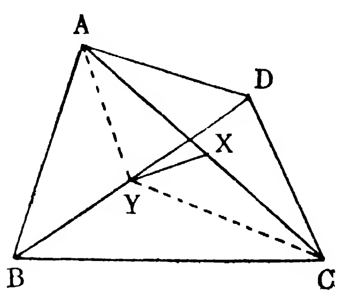
$\therefore$ $AP=CP$ and $BP=DP$.

Now, in the $\triangle ABC$, $AB^2+BC^2=2(AP^2+BP^2)$
and in the $\triangle ADC$, $CD^2+DA^2=2(AP^2+DP^2)$ } [Apollonius' Theorem]

$\therefore$ by adding these two results,

$$\begin{aligned} AB^2+BC^2+CD^2+DA^2 &= 4AP^2+2BP^2+2DP^2 \\ &= 4AP^2+4BP^2 \quad [\because BP=DP] \\ &= (2AP)^2+(2BP)^2 \\ &= AC^2+BD^2. \end{aligned}$$

Ex. 6. *Prove that in any quadrilateral the sum of the squares on its sides is equal to the sum of the squares on its diagonals together with four times the square on the straight line joining the middle points of the diagonals.*



Let the middle points of the diagonals AC, BD of the quadrilateral $ABCD$ be X, Y respectively. Join XY.

It is reqd. to prove that

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4XY^2.$$

Construction. Join AY, CY.

Proof. In the $\triangle ABD$, $AB^2+AD^2=2(BY^2+AY^2)$
and in the $\triangle BCD$, $BC^2+CD^2=2(BY^2+CY^2)$ } [Apollonius' Theo.]

$\therefore$ by adding $AB^2+AD^2+BC^2+CD^2=4BY^2+2(AY^2+CY^2)$.

Again in the $\triangle ACY$, $AY^2 + CY^2 = 2(AX^2 + 2XY^2)$ [Apollonius' Theorem]

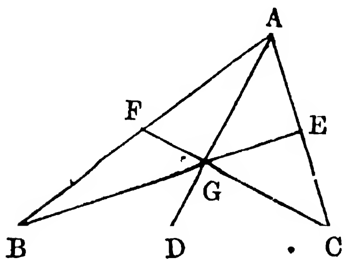
$\therefore \quad AB^2 + BC^2 + CD^2 + DA^2 = 4BY^2 + 4AX^2 + 4XY^2$

$$= (2BY)^2 + (2AX)^2 + 4XY^2$$

$$= BD^2 + AC^2 + 4XY^2.$$

Ex. 7. *If G is the centroid of a triangle ABC, prove that* $AB^2 + AC^2 + BC^2 = 3(GA^2 + GB^2 + GC^2)$.

[The point of intersection of the three medians of a triangle is called the **Centroid** of the triangle.]



Let G be the pt. of intersection of the three medians AD, BE, CF of the $\triangle ABC$, so that G is the centroid of the $\triangle$.

It is reqd. to prove that

$$AB^2 + AC^2 + BC^2 = 3(GA^2 + GB^2 + GC^2).$$

Proof. By Apollonius' Theorem, we have

$$AB^2 + AC^2 = 2(BD^2 + AD^2)$$

$$BC^2 + AC^2 = 2(AF^2 + CF^2)$$

and $$AB^2 + BC^2 = 2(AE^2 + BE^2).$$

Adding these results, we get

$$2(AB^2 + BC^2 + AC^2) = 2(BD^2 + AF^2 + AE^2) + 2(AD^2 + BE^2 + CF^2)$$

that is, $4(AB^2 + BC^2 + AC^2) = 4(BD^2 + AF^2 + AE^2) + 4(AD^2 + BE^2 + CF^2)$

$$= (2BD)^2 + (2AF)^2 + (2AE)^2 + 4(AD^2 + BE^2 + CF^2)$$

$$= BC^2 + AB^2 + AC^2 + 4(AD^2 + BE^2 + CF^2).$$

$\therefore \quad 3(AB^2 + BC^2 + AC^2) = 4(AD^2 + BE^2 + CF^2).$

Since G is the pt. of trisection of the medians, $GA = \frac{2}{3}AD$, $GB = \frac{2}{3}BE$ and $GC = \frac{2}{3}CF$.

$\therefore \quad 4AD^2 = 9GA^2$, $4BE^2 = 9GB^2$ and $4CF^2 = 9GC^2$.

$\therefore \quad 4(AD^2 + BE^2 + CF^2) = 9(GA^2 + GB^2 + GC^2).$

Hence, $3(AB^2 + BC^2 + AC^2) = 9(GA^2 + GB^2 + GC^2)$.

i.e. $AB^2 + BC^2 + AC^2 = 3(GA^2 + GB^2 + GC^2)$.

Ex. 8. *Prove that the locus of a point, which moves so that the sum of the squares of its distances from two fixed points is constant, is a circle.*

Let A and B be two fixed pts. and P a moving point, such that $PA^2 + PB^2 =$ a constant.

It is reqd. to prove that the locus of P is a circle.

Construction. Join AB and let O be the middle pt. of AB.

Proof. By Apollonius' Theorem,

$$PA^2+PB^2=2(AO^2+PO^2)=\text{a constant.}$$

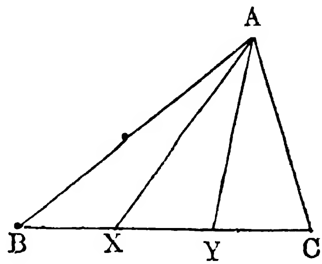
But A and B being fixed pts., the middle pt. O of AB is also a fixed pt.

$\therefore$ AO = a constant.

Again, since AO^2+PO^2 is constant (proved), PO must also be constant, *i.e.* the distance of the moving pt. P from the fixed pt. O is always the same.

Hence, the locus of P is a circle with centre at O, the middle pt. of AB.

Ex. 9. *In a triangle ABC, X is a fixed point of trisection of the base BC adjacent to the angle B; prove that $2AB^2+AC^2=6BX^2+3AX^2$.*



Let X be a pt. in the base BC of a triangle ABC such that $BX=\frac{1}{3}BC$. Join AX.

It is reqd. to prove that

$$2AB^2+AC^2=6BX^2+3AX^2.$$

Construction. Bisect XC at Y. Join AY.

Then $BX=XY=YC$.

Proof. Now, in the $\triangle ABY$, $AB^2+AY^2=2(BX^2+AX)^2$ [Apollonius Theorem]

$\therefore$ $2AB^2+2AY^2=4BX^2+4AX^2$.

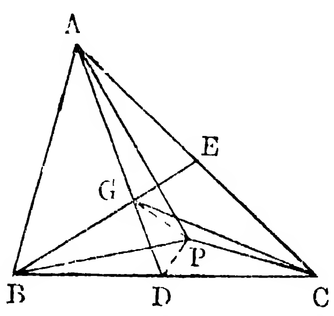
Again, in the $\triangle AXC$, $AX^2+AC^2=2(XY^2+AY^2)$ [Apollonius' Theorem]

Adding the results, we have

$$2AB^2+2AY^2+AX^2+AC^2=4BX^2+4AX^2+2XY^2+2AY^2.$$

$$\therefore\ 2AB^2+AC^2=6BX^2+3AX^2, \text{ since } BX=XY.$$

Ex. 10. *If G is the centroid of a triangle ABC and P any point, prove that $PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3GP^2$.*



Let G be the pt. of intersection of the two medians AD, BE of the $\triangle ABC$ and P any point. Then G is the centroid of the $\triangle ABC$. Join PA, PB, PC, GP and GC.

It is reqd. to prove that $PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3GP^2$.

Construction. Join PD.

Proof. Since $AG=2GD$, with the help of the preceding Ex. 9, from the $\triangle APD$, we get $2PD^2+PA^2=6GD^2+3GP^2$.

Now, $PB^2+PC^2=2BD^2+2PD^2$. [Apollonius' Theorem]

Adding these two results, we have

$$2PD^2+PA^2+PB^2+PC^2=6GD^2+3GP^2+2BD^2+2PD^2$$

i.e.

$$\begin{aligned} PA^2+PB^2+PC^2 &= 6GD^2+2BD^2+3GP^2 \\ &= 4GD^2+2(GD^2+BD^2)+3GP^2 \\ &= GA^2+GB^2+GC^2+3GP^2. \end{aligned}$$

[$\because$ $GA=2GD$].

N.B. There will be no change in the proof, if the point P be taken outside the circle.

Exercise I

1. ABC is an isosceles triangle in which $AB=AC$, and CD is drawn perpendicular to AB; prove that $BC^2=2AB.BD$.

2. The angle ACB of a triangle ABC is 120°; prove that

$$AB^2=AC^2+BC^2+AC.BC.$$

3. Show that a triangle whose sides are 17 cm., 13 cm. and 11 cm. is an acute-angled triangle.

4. The sides of a triangle are 13, 35 and 43. Show that it is an obtuse-angled triangle, and that the obtuse angle is 120°.

5. The base of a triangle is 12 cm. and the sum of the squares on the other two sides is 234 sq. cm.; find the locus of the vertex.

6. The base BC of an isosceles triangle is produced to D making $CD=BC$; prove that $AD^2=AB^2+2BC^2$.

7. O is the point of intersection of the diagonals of a square $ABCD$ and P is any other point. Prove that

$$AP^2+BP^2+CP^2+DP^2=4OA^2+4OP^2.$$

8. ABC is an acute-angled triangle and AD, BE and CF are perpendiculars to the opposite sides BC, CA and AB; show that $AB^2+BC^2+CA^2=2(AB.AF+BC.BD+AC.CE)$.

9. If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals, prove that the quadrilateral is a parallelogram.

10. Prove that in any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the middle points of the opposite sides.

11. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians. Prove this.

12. ABC is an isosceles triangle and A is joined to any point D in the base BC or BC produced; DA is produced to E making $AE = AD$; show that the difference of the squares on BE and CE is equal to the difference of the squares on BD and CD.

13. The sum of the squares on the diagonals of a trapezium is equal to the sum of the squares on the two non-parallel sides together with twice the rectangle contained by the two parallel sides. Prove this.

14. In any triangle prove that the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertex to the base.

CHAPTER II

RATIO AND PROPORTION

2·1. Ratio and Proportion.

The number which expresses the relation which one quantity bears to another of the same kind is called the **Ratio** of the two quantities. The ratio expresses what multiple or part one quantity is of the other.

To determine what multiple or what part is one straight of another in length, we divide the length of the first by the length of the second, their length being expressed in the same unit. In a similar manner to find how many times a quadrilateral is of a triangle in area, we divide the area of the quadrilateral by the area of the triangle, both expressed in the same unit. Hence, the ratio is an abstract number and not a concrete one.

Suppose there are two straight lines 12 cms. and 8 cms. long. To determine how many times the first of them is greater than the second, we divide 12 cms. by 8 cms. and get the absolute number $\frac{12}{8}$ or $\frac{3}{2}$. Hence the ratio of the lengths of these two straight lines is $\frac{3}{2}$ and this is usually written as $3 : 2$. If the measure of two quantities of the same kind be 'a' units and 'b' units respectively, we determine their ratio by dividing 'a' units by 'b' units, which is usually written as $a \div b$. By omitting the '–' from the sign of division '$\div$' we write ':' as the sign of a ratio.

With the help of a ratio we compare two quantities of the same kind. As two quantities cannot be compared unless they are of the same kind, so the ratio of two quantities cannot be ascertained unless they are of the same kind. Again, if two quantities of the same kind are expressed in different units; to compare the one with the other both of them have got to be expressed in the same unit. For example, there are two straight lines 4 ft. and 8 in. long. Their ratio will not be denoted by $4 : 8$ or $1 : 2$. Both of them have to be reduced either to feet or to inches and then the ratio of the two has to be determined. Hence the ratio of the lengths of these two straight lines will be denoted by $48 : 8$ when expressed in inches or by $4 : \frac{2}{3}$ when expressed in feet and both of them will give the same result $6 : 1$.

The two quantities 'a' and 'b' of the ratio $a : b$ are called the terms of the ratio. The first term 'a' is called the **antecedent** and the second term 'b' is called the **consequent.**

Commensurable and Incommensurable Quantities

If two quantities can be expressed as multiples of some common factor as unit *i.e.* can be completely expressed by the ratio of two integers, the quantities are said to be **commensurable** to each other ; and quantities that cannot be so expressed are said to be **incommensurable** quantities.

The two quantities 6 ft. and 10 ft. are commensurable to each other as the ratio of these two quantities can be expressed by the ratio of the two integers 6 and 10. If the length of the side of a square is 1 cm., then the length of the diagonal is $\sqrt{2}$ cms. Hence, the sides of the square is not commensurable with the diagonal as the ratio of the two $1 : \sqrt{2}$ cannot be expressed by two integers ; or in other words, the sides of a square and its diagonal are incommensurable quantities.

Proportion

Four quantities are said to be **proportionals** or in **proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth, or in other words when two ratios are equal, the four quantities forming the two ratios are called proportionals. Suppose, the ratio of 'a' to 'b' is equal to the ratio of 'c' to 'd', then the four quantities, a, b, c, d are proportionals or in proportion. This is usually expressed by $a : b = c : d$ or $a : b : : c : d$.

Of the four quantities a, b, c, d forming the two ratios, the first 'a' and the fourth 'd' are called the **extremes** and the two middle ones 'b' and 'c' are called the **means.** The fourth quantity 'd' is called the **fourth proportional** to **a, b, c.**

If three quantities a, b, c be such that the ratio of the first to the second is equal to the ratio of the second to the third, *i.e.* if $a : b = b : c$, then the three quantities a, b, c are said to be in *continued proportion*, the second quantity 'b' is called the **mean proportional** between 'a' and 'c' and the third quantity 'c' is called the **third proportional** to 'a' and 'b'.

Ratio being an absolute number, all the four quantities forming the two ratios of a proportion may be of one kind or the first two quantities forming the first ratio may be of one kind and the remaining two quantities forming the second ratio of a different kind. This will be made clear by an example. Without committing any error, we can say that the ratio of Rs. 10 to Rs. 5 is equal to the ratio of 6 hrs. to 3 hrs., since both of them are equal to 2 : 1. Hence, we can rightly say that Rs. 10, Rs. 5, 6 hrs. and 3 hrs. are in proportion.

Some Important Fundamental Theorems on Proportion.

If four quantities a, b, c, d are in proportion, several other useful proportions can be derived from this by making suitable operations on the given proportion. The method applied to arrive at a particular result is given in any book on Algebra. In Geometry we are only concerned with the final result obtained after the operation. The different results go by different names as given below without showing how they are obtained.

(i) If a, b, c, d be in proportion *i.e.* $a : b = c : d$, then

(1) $ad = bc$; this operation is known as **Cross-multiplication** ;

(2) $\frac{a}{c} = \frac{b}{d}$; this operation is known as **Alternando** ;

(3) $\frac{b}{a} = \frac{d}{c}$; this operation is known as **Invertendo** ;

(4) $\frac{a+b}{b} = \frac{c+d}{d}$; this operation is known as **Componendo** ;

(5) $\frac{a-b}{b} = \frac{c-d}{d}$; this operation is known as **Dividendo** ;

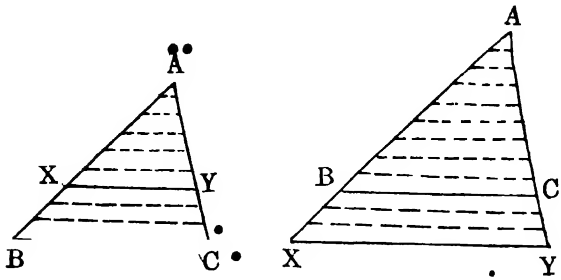
and (6) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$; this operation is known as **Componendo-Dividendo.**

(ii) If $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \cdots$, then each of these ratios

$$= \frac{a+b+c+\cdots}{x+y+z+\cdots}$$; this operation is known as **Addendo.**

Theorem 3

If a straight line is drawn parallel to one side of a triangle, it divides the other two sides or those sides produced proportionately.



Let the straight line XY be drawn $\parallel$ to the side BC of the $\triangle ABC$ to cut the sides AB and AC or those sides produced at X and Y respectively.

It is required to prove that $AX : XB = AY : YC$.

Proof. Let the str. line XY be drawn $\parallel$ to the side BC of the $\triangle ABC$ to cut the sides AB and AC, or those sides produced, at X and Y respectively.

Let the str. lines AX and XB be commensurable to each other and let m be their common measure. Then each of the str. lines AX and XB can be divided into some equal parts, each equal to 'm'. Suppose AX is divided into p equal parts, and XB into q equal parts, each equal to 'm'.

$\therefore \quad AX = pm$ and $XB = qm$.

$\therefore \quad AX : XB = pm : qm = p : q$.

Now, through the pts. of division in AX and XB draw str. lines $\parallel$ to the side BC. These $\parallel$ str. lines will divide AY and YC into p and q equal parts respectively.

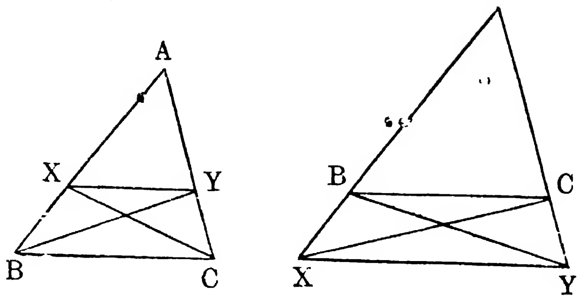
Let the length of each of these equal parts be n.

Then $AY = pn$ and $YC = qn$.

$\therefore \quad AY : YC = pn : qn = p : q$.

$\therefore \quad AX : XB = AY : YC$.

Alternative Method.



Construction. Join BY, CX.

Proof. Since BC and XY are parallel,

$\triangle BXY = \triangle CXY$ [on the same base XY]

$\therefore$ $\triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY$.

Now, the areas of two $\triangle^s$ of equal altitudes are to one another as their bases.

$\therefore$ $\triangle AXY : \triangle BXY = AX : XB$. [the vertex being the same
and $\triangle AXY : \triangle CXY = AY : YC$. in the two cases, the $\triangle^s$
$\therefore$ $AX : XB = AY : YC$. are of equal altitudes.]

Corollary. If the str. line XY drawn $\parallel$ to the side BC cuts AB, AC at X, Y respectively, then $AB : AX = AC : AY$.

Now, by Theorem 3, $\dfrac{AX}{XB} = \dfrac{AY}{YC}$.

$\dfrac{XB}{AX} = \dfrac{YC}{AY}$ [by Invertendo]

$\therefore$ in Fig. 1, $\dfrac{XB + AX}{AX} = \dfrac{YC + AY}{AY}$ [by Componendo]

and in Fig. 2, $\dfrac{AX - XB}{AX} = \dfrac{AY - YC}{AY}$ [by Dividendo]

$\dfrac{AB}{AX} = \dfrac{AC}{AY}$, *i.e.* $AB : AX = AC : AY$.

N. B. In a similar way it can be proved in this case that

$AB : BX = AC : CY$.

Theorem 4

[CONVERSE OF THEOREM 3]

If a straight line cuts two sides of a triangle, or those two sides produced, proportionately, then it is parallel to the remaining side of the triangle.

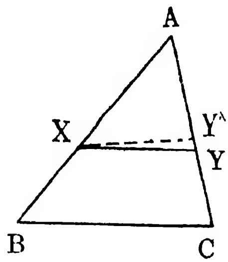


Fig. 1

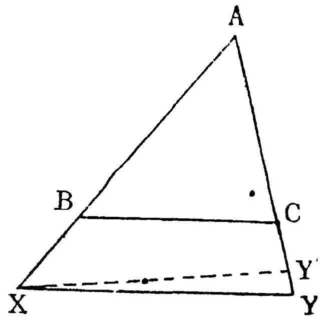


Fig. 2

In the $\triangle ABC$ let the str. line XY be drawn to cut the sides AB and AC in X and Y respectively, so that $AX : XB = AY : YC$.

It is reqd. to prove that the str. line XY is $\parallel$ to the side BC.

Proof. If the str. line XY be not $\parallel$ to BC, let XY' be $\parallel$ to BC and let it cut the side AC in Y'.

$\therefore \quad AX : XB = AY' : Y'C.$

But, by hypothesis, $AX : XB = AY : YC$.

$\therefore \quad AY' : Y'C = AY : YC.$

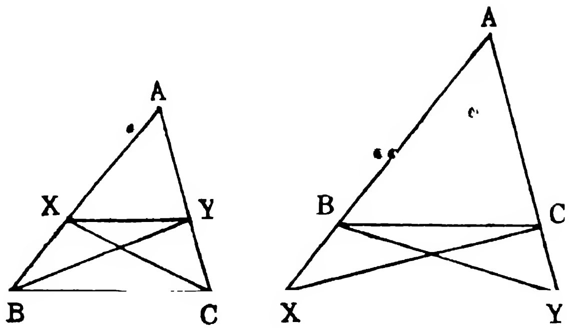
$\therefore$ by Componendo in Fig. 1 and by Dividendo in Fig. 2,

$$AC : Y'C = AC : YC.$$

$\therefore \quad YC = Y'C$ *i.e.* the two points Y, Y' coincide with each other.

the str. line XY is $\parallel$ to BC.

Alternative Method.



Construction. Join BY, CX.

Proof. Since the vertex Y of the $\triangle AXY$ and the $\triangle BXY$ is the same, they are of equal altitudes.

$\therefore \quad \triangle AXY : \triangle BXY = AX : XB.$

Similarly, $\triangle AXY : \triangle CXY = AY : YC.$

But $AX : XB = AY : YC.$

$\therefore \quad \triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY.$

$\therefore \quad \triangle BXY = \triangle CXY.$

But these two $\triangle^s$ stand on the same base XY and on the same side of it. $\therefore$ they must be between the same parallels

i.e. XY is $\parallel$ to BC.

Corollary. If X, Y be two pts. in the sides AB, AC of a $\triangle ABC$, so that $AB : AX = AC : AY$ (or $AB : BX = AC : CY$), then the str. line XY is $\parallel$ to the base BC.

The proof is similar to that of the Corollary to Theorem 3.

Exercise II(A)

1. Three parallel straight lines cut any two transversals proportionally. Prove this.

2. Prove that the straight line which joins the middle points of two sides of a triangle is parallel to the third side.

3. Prove that the straight line drawn through the middle point of one of the sides of a triangle parallel to another side bisects the remaining side.

4. Three straight lines OAP, OBQ and OCR are such that AB is parallel to PQ and BC to QR; show that AC is parallel PR.

5. ABC is a triangle and through D, a point in AB, DE is drawn parallel to BC cutting AC in E; CF is drawn parallel to EB meeting AB produced in F. Prove that AB is a mean proportional between AD and AF.

6. Prove that the line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

[*Hints : Produce the oblique sides to meet at a point.*]

7. From any point in the base of a triangle, straight lines are drawn parallel to the sides of the triangle; show that the point of intersection of the diagonals of every parallelogram so formed lies on a straight line parallel to the base.

8. ABC and ABD are two triangles and M and N are the iddle points of BC and BD respectively; prove that MN is parallel to the join of the centroid of the triangles.

9. From a point E in the common base of two triangles ACB, ADB straight lines are drawn parallel to AC, AD meeting BC, BD at F, G; show that FG is parallel to CD.

10. E is the middle point of the median AD of a triangle ABC and BE is produced to cut AC in F; prove that $AF = \frac{1}{3}AC$.

[*Hints : Draw $DG \parallel$ to BF.*]

11. OPQ is a straight line drawn through a fixed point O and is such that $OP : OQ$ is constant. If P moves along a fixed straight line, find the locus of Q.

2·2. Equiangular Triangles and Polygons.

Two triangles or two polygons are said to be equiangular to each other, if the angles of the one are equal to the angles of the other, when taken in the same order.

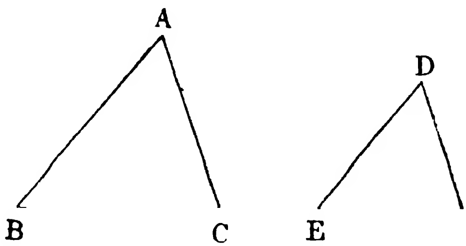
In equiangular triangles the sides opposite to the equal angles are called **corresponding sides**; and in equiangular polygons

the sides adjacent to two successive equal angles are called corresponding sides.

Similar Triangles and Polygons.

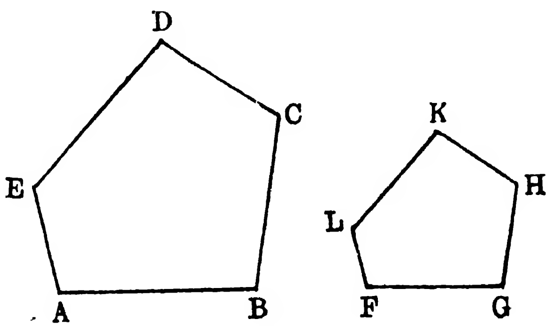
If two equiangular triangles or polygons have their corresponding sides proportional, they are said to be **similar.**

It will be seen later on that two equiangular triangles are always similar. But two equiangular polygons may or may not be similar.



In the two triangles ABC and DEF given above, if $\angle A = \angle D$ $\angle B = \angle E$, $\angle C = \angle F$ and if $AB : DE = BC : EF = CA : FD$, the two $\triangle^s$ ABC, DEF will be similar.

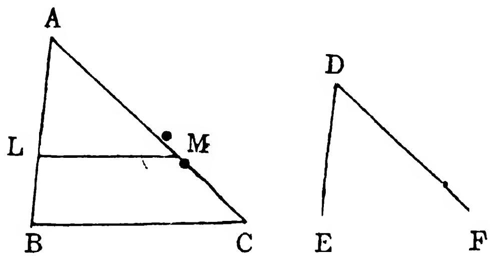
Again in the two polygons $ABCDE$ and $FGHKL$ given here, if $\angle A = \angle F$, $\angle B = \angle G$, $\angle C = \angle H$, $\angle D = \angle K$, $\angle E = \angle L$ and if $AB : FG = BC : GH = CD : HK = DE : KL = EA : LF$, then the two polygons $ABCDE$, $FGHKL$ are similar. Since $\angle A = \angle F$ and $\angle B = \angle G$, the two sides AB and FG being adjacent to these equal angles are corresponding sides of the two polygons ; for similar reasons BC and GH are correponding sides, etc.



N. B. The sum of the three angles of a triangle being two right angles, if two angles of a triangle be equal to two angles of another, the two triangles will be equiangular.

Theorem 5

If two triangles are equiangular, their corresponding sides are proportional.



In the Δ^s ABC and DEF, let $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

It is required to prove that $AB : DE = BC : EF = CA : FD$.

Proof. From AB, AC cut off AL, AM equal to DE, DF respectively and join LM.

Then in the Δ^s ALM and DEF,

$AL = DE$, $AM = DF$ and $\angle A = \angle D$. (Hyp.)

$\therefore$ the Δ^s ALM and DEF are equal in all respects.

$\therefore$ $\angle L = \angle E = \angle B$, $\therefore$ LM is $\parallel$ to BC.

$\therefore$ $AB : AL = AC : AM$ *i.e.* $AB : DE = AC : DF$.

Similarly, it may be proved that $BC : EF = AC : DF$.

$\therefore$ $AB : DE = BC : EF = AC : DF$.

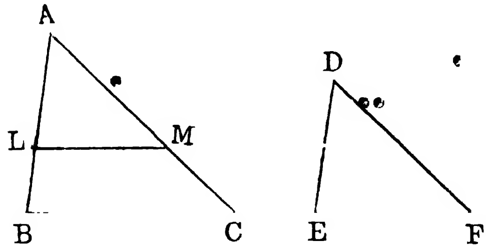
Theorem 6

[CONVERSE OF THEOREM 5]

If two triangles have their sides proportional, when taken in order, they are equiangular, equal angles being opposite to corresponding sides.

Let ABC and DEF be two $\triangle^s$ in which

$$AB : DE = BC : EF = CA : FD.$$



It is reqd. to prove that the $\angle^s$ A, B, C opposite to the sides BC, CA, AB are respectively equal to the $\angle^s$ D, E, F opposite to the corresponding sides EF, FD, DE.

Construction. From AB, AC cut off AL, AM equal respectively to DE, DF. Join LM.

Proof. Since $AB : DE = AC : DF$ (Hyp.) and, by construction,

$$DE = AL \text{ and } DF = AM,$$

$\therefore$ $AB : AL = AC : AM.$

$\therefore$ LM is $\parallel$ to BC.

$\therefore$ the $\angle ALM =$ the corresponding $\angle ABC$ and the $\angle AML$ $=$ the corresponding $\angle ACB$.

$\therefore$ the $\triangle^s$ ALM and ABC are equiangular.

$\therefore$ $AB : AL = BC : LM$, that is $AB : DE = BC : LM$.

But $AB : DE = BC : EF$ (Hyp.). $\therefore$ $LM = EF$.

And $AL = DE$ and $AM = DF$ (by construction).

$\therefore$ the $\triangle^s$ ALM and DEF are equal in all respects.

$\therefore$ the $\angle ALM =$ the $\angle DEF$ and the $\angle AML =$ the $\angle DFE$.

But the $\angle ALM =$ the $\angle B$ and the $\angle AML =$ the $\angle C$.

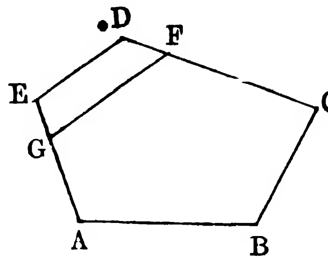
$\therefore$ the $\angle B =$ the $\angle E$ and the $\angle C =$ the $\angle F$.

$\therefore$ the remaining $\angle A =$ the remaining $\angle D$.

$\therefore$ the $\triangle^s$ ABC and DEF are equiangular.

N. B. In Theorem 5, it has been proved that when two triangles are equiangular, they are similar. And it has also been pointed out that two equiangular polygons may not be similar.

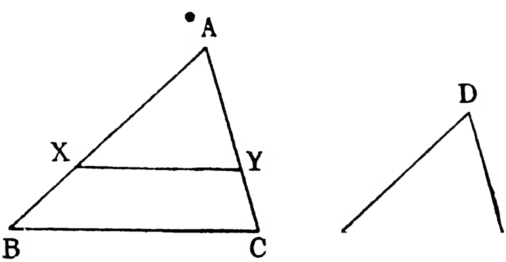
Suppose $ABCDE$ is a polygon. Through a point F in CD draw $FG \parallel$ to DE to cut the side EA in G. It is clear from the figure that the two polygons $ABCDE$ and $ABCFG$ are equiangular, but their corresponding sides can never be proportional. If they were proportional, then $BC : CF$ would have been equal to $BC : CD$ *i.e*, CF would have been equal to CD, which is impossible.



In order that two polygons may be similar, two conditions are to be satisfied (1) the two polygons should be equiangular and (2) their corresponding sides should also be proportional. But in the case of two triangles, one of the two conditions will be sufficient to make them similar. Hence, if two triangles are equiangular or their corresponding sides are proportional, they will be similar.

Theorem 7

If two triangles have one angle of the one equal to one angle of the other and the sides about the equal angles proportional, the triangles are similar.



In the Δ^s ABC and DEF, let the $\angle A =$ the $\angle D$ and $AB : DE = AC : DF$.

It is required to prove that Δ^s ABC and DEF are similar.

Construction. From the sides AB and AC of the ΔABC cut off AX and AY respectively equal to DE and DF. Join XY.

Proof. In the Δ^s AXY and DEF, $AX = DE$, $AY = DF$ and the included $\angle A =$ the included $\angle D$.

$\therefore$ the Δ^s AXY and DEF are equal in all respects.

$\therefore$ $\angle AXY = \angle E$ and $\angle AYX = \angle F$.

Now, since $AB : DE = AC : DF$. $\therefore$ $AB : AX = AC : AY$.

$\therefore$ XY is $\parallel$ to BC.

$\therefore$ the $\angle AXY =$ the corresponding $\angle B$ and the $\angle AYX =$ the corresponding $\angle C$.

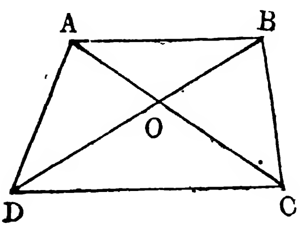
But the $\angle AXY =$ the $\angle E$ and the $\angle AYX =$ the $\angle F$.

$\therefore$ the $\angle B =$ the $\angle E$ and the $\angle C =$ the $\angle F$.

the Δ^s ABC and DEF are equiangular, and hence similar.

2·3. Illustrative Examples

Ex. 1. *AB and DC are the parallel sides of a trapezium whose diagonals AC and BD intersect at O ; prove that* $AB : CD = OA : OC = OB : OD$.



Let the diagonals AC, BD of the trapezium with the sides AB, DC parallel intersect at O.

It is reqd. to prove that

$AB : CD = AO : OC = OB : OD$.

Proof. In the Δ^s AOB, COD, $\angle OAB = \angle OCD$ and $\angle OBA = \angle ODC$.

[$\because$ AB is $\parallel$ to DC]

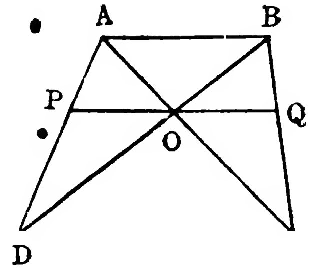
$\therefore$ the Δ^s AOB, COD are equiangular, and hence similar.

$\therefore$ $AB : CD = OA : OC = OB : OD$.

Ex. 2. *Prove that a line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at the point.*

Let the diagonals AC, BD of the trapezium $ABCD$ intersect at O. Let the straight line POQ drawn through O parallel to AB or CD cut the sides AD, BC at P, Q respectively.

It is required to prove that $OP=OQ$.



Proof. In the $\triangle^s$ OAB and OCD, $\angle OAB=\angle OCD$ and $\angle OBA=\angle ODC$

[$\because$ AB is $\parallel$ to DC]

$\therefore$ the $\triangle^s$ OAB and OCD are equiangular and hence similar.

$\therefore$ $OA : OC=OB : OD$.

$\therefore$ $(OA+OC) : OC=(OB+OD) : OD$.

[by Componendo]

i.e. $AC : OC=BD : OD$.

Again, since OP is $\parallel$ to AB, the $\triangle^s$ DPO and DAB are similar.

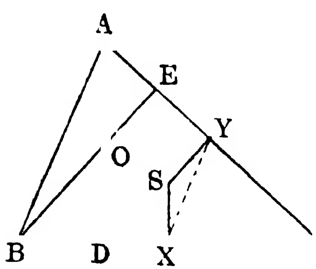
$\therefore$ $BD : OD=AB : OP$.

In a similar way, $AC : OC=AB : OQ$.

But $AC : OC=BD : OD$ (*proved*),

$\therefore$ AB $OP=AB : OQ$; that is, $OP=OQ$.

Ex. 3. *In any triangle, prove that the distance of the ortho-centre from any angular point is double the distance of the circum-centre from the opposite side.*



Let the pt. of intersection of AD, BE, perpendiculars drawn to BC, CA respectively, be O, the ortho-centre and the pt. of intersection of the perps. drawn to BC, CA from their middle points X, Y be S, the circum-centre of the $\triangle$ ABC

It is reqd. to prove that $OA=2SX$.

Construction. Join XY.

Proof. Since X, Y are the middle points of BC, CA respectively,

$\therefore$ XY is $\parallel$ to AB and $=\frac{1}{2}AB$.

Now, SX is $\parallel$ to AD, both being perps. to BC,

and SY is $\parallel$ to BE, both being perps. to CA.

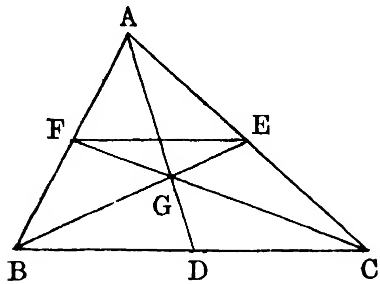
$\therefore$ the two $\triangle^s$ SXY and OAB are equiangular, and hence similar.

$\therefore$ $OA : SX=AB : XY=2 : 1$.

i.e. $OA=2SX$.

Ex. 4. *Prove that the two medians of a triangle intersect at their point of trisection. Hence, show that the medians of a triangle are concurrent.*

Let E and F be the middle points of the sides AC and AB of the $\triangle ABC$ and let the medians BE and CF intersect at G and let D be the middle point of the side BC.



It is required to prove that

(i) $GE=\frac{1}{3}BE$ and $GF=\frac{1}{3}CF$ and

(ii) the three medians of the $\triangle ABC$ are concurrent.

Construction. Join EF.

Proof. Since E and F are middle points of AC and AB, $FE \parallel$ to BC and $=\frac{1}{2}BC$ and, $\therefore$ the $\triangle$s FGE and BGC are equiangular, and hence similar,

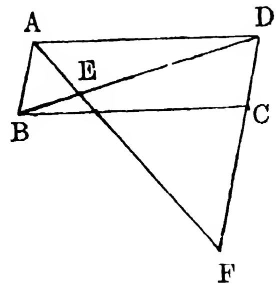
$\therefore \quad BG : GE = CG : GE = BC : FE = 2 : 1.$

$\therefore \quad GE=\frac{1}{3}BE$ and $GF=\frac{1}{3}CF.$

In a similar way, if the medians AD and BE intersect at G', it may be shown that $G'E=\frac{1}{3}BE$, *i.e.* the two pts. G and G' are coincidental and $G'D=\frac{1}{3}AD$ *i.e.*, $GD=\frac{1}{3}AD$,

Ex. 5. *The diagonal BD of a parallelogram $ABCD$ is divided at E so that BE is one-third of ED and AE, DC is produced to meet at F; show that $FC=2AB$.*

Let E be a point on the diagonal BD of the parallelogram $ABCD$ such that $BE=\frac{1}{3}ED$; let the join of A and E be produced to meet DC produced in F.



It is reqd. to prove that $FC=2AB$.

Proof. Since AB and DF are parallel, the two $\triangle$s ABE and DFE are equiangular, and hence similar.

$\therefore \quad AB : DF = BE : ED = 1 : 3.$

$\therefore \quad DF = 3AB.$

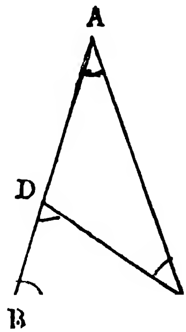
But $AB=DC$, being the opposite sides of a parallelogram.

$\therefore \quad FC=2AB.$

Ex. 6. *ABC is an isosceles triangle having each of the angles at the base double of the vertical angle BAC; the bisector of the angle ACB meets AB at D. Show that AB, BC, BD are proportionals.*

Let ABC be an isosceles $\triangle$ in which $AB=AC$ and each of the angles ABC and ACB at the base is double of the vertical $\angle BAC$; let CD, the bisector of the $\angle ACB$ meet AB at D.

It is reqd. to prove that AB, BC and BD are in continued proportion *i.e.* $AB : BC = BC : BD$.



Proof. Since the $\angle ABC =$ the $\angle ACB = 2\angle BAC$, and the sum of the $\angle^s$ of the $\triangle ABC = 180°$,

$\therefore$ the $\angle ABC =$ the $\angle ACB = 72°$ and the $\angle BAC = 36°$.

Again, CD is the bisector of the $\angle ACB$,

$\therefore$ the $\angle BCD = 36°$.

$\therefore$ in the $\triangle BCD$, the $\angle BCD = 36°$ and the $\angle DBC$ *i.e.* $\angle ABC = 72°$.

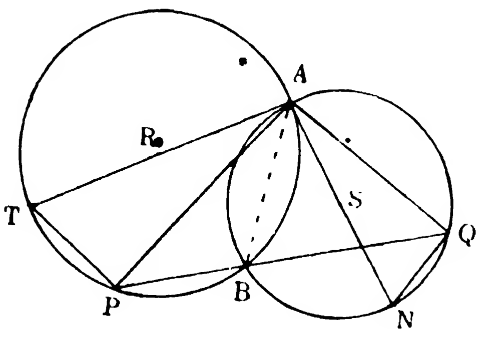
$\therefore$ the remaining $\angle BDC = 72°$.

$\therefore$ the $\triangle^s$ ABC and CBD are equiangular, and hence similar.

$\therefore$ $AB . BC = BC : BD$.

i.e. AB, BC are in continued proportion.

Ex. 7. *Two circles intersect at A and B, and through B any straight line PBQ is drawn to cut the two circles at P and Q. Prove that their diameters are as AP : AQ.*



Let the two circles whose centres are at R and S intersect at A and B, and through B let PBQ be drawn to cut the two circles at P and Q. Join AP and AQ ; let ART and ASN be diameters of the two circles.

It is reqd. to prove that $AT : AN = AP : AQ$.

Proof. Join AB, TP and NQ.

Now, the $\angle ANQ$ = the $\angle ABQ$, (being in the same segment)

Again, $ABPT$ is a cyclic quadrilateral, $\therefore$ the ext. $\angle ABQ$ = the opposite int. $\angle ATP$. $\therefore$ the $\angle ANQ$ = the $\angle ATP$.

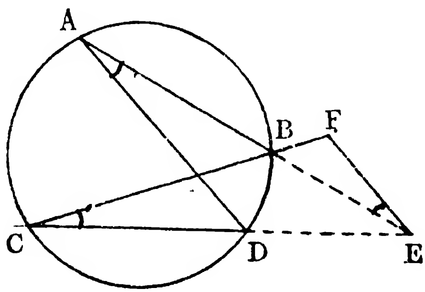
In the $\triangle^s$ APT and AQN,

the $\angle ATP$ = the $\angle ANQ$ (proved) and the $\angle APT$ = the $\angle AQN$ (being in semi-circles).

$\therefore$ the triangles are equiangular, and hence similar.

$\therefore$ $AT : AN = AP : AQ$.

Ex. 8. *Chords AB and CD of a circle are produced to meet at E, and EF is drawn through E parallel to AD meeting CB produced in F. Prove that EF is a mean proportional between FB and FC.*



Let the two chords AB, CD of a circle be produced to meet at E and through E let EF be drawn $\parallel$ to AD to meet CB produced at F.

It is reqd. to prove that EF is a mean proportional between FB and FC.

Proof. In the $\triangle^s$ EFB and EFC, the $\angle EFB$ is common and the $\angle ECF$
= the $\angle BAD$ (in the same segment)
= the $\angle BEF$ [$\because$ EF is $\parallel$ to AD]

$\therefore$ the two $\triangle^s$ EFB and EFC are equiangular, and hence similar.

$\therefore$ $FB : FE = FE : FC$.

i.e. FE is a mean proportional between FB and FC.

Ex. 9. *A common tangent to two circles cuts their line of centres internally or externally in the ratio of their radii. Prove this.*

Let O and O' be the centres of the two circles and PT their transverse common tangent in Fig. 1 and direct common tangent in Fig. 2 and let the tangent cut the line of centres (i) in Fig. 1, and (ii) OO' produced in Fig 2, at R.

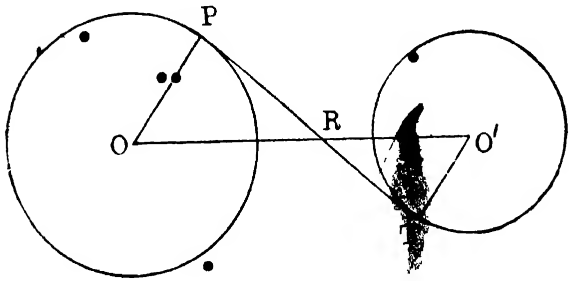


Fig. 1

It is required to prove (in both Figs.) that $OR : O'R = OP : O'T$.

Proof. Join OP and $O'T$.

In the $\triangle^s$ OPR and $O'TR$, the $\angle OPR =$ the $\angle O'TR$ (the radius through the point of contact being perp. to the tangent, each is a rt. $\angle$) and the $\angle ORP =$ the $\angle O'RT$ (vertically opp. $\angle^s$ in Fig. 1, and the same $\angle$ in Fig. 2.)

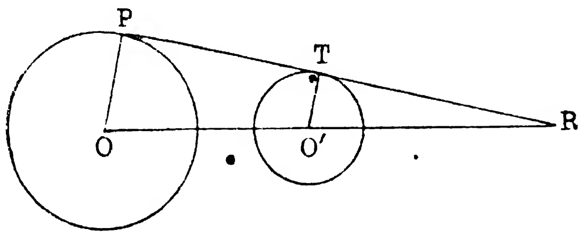


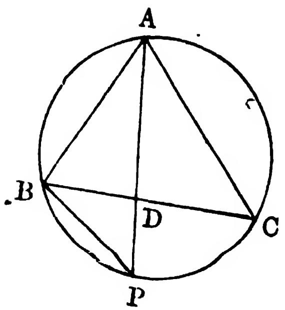
Fig. 2

$\therefore$ the $\triangle^s$ OPR and $O'TR$ are equiangular, and hence similar.

$\therefore$ $OR : O'R = OP : O'T$.

Ex. 10. *If the internal bisector of the vertical angle of a triangle meets the base, prove that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the bisector.*

In the $\triangle ABC$ let AD, the bisector of the vertical $\angle BAC$ meet the base BC at D.



It is reqd. to prove that

$$AB \cdot AC = BD \cdot DC + AD^2.$$

Construction. Draw the circum-circle of the $\triangle ABC$ and produce AD to meet the circum-circle in P.

Proof. Join BP.

In the $\triangle^s$ ADC and BDP, the $\angle CAD$ = the $\angle PBD$ and the $\angle ACD$ = the $\angle BPD$ (in the same segment).

$\therefore$ the $\triangle^s$ ADC and BDP being equiangular are similar.

$\therefore$ $BD : DP = AD : DC$ i.e., $AD \cdot DP = BD \cdot DC$.

Now, in the $\triangle^s$ BAP and DAC, the $\angle BAP$ = the $\angle DAC$ (Hyp.)
and the $\angle BPA$ = the $\angle DCA$ (in the same segment).

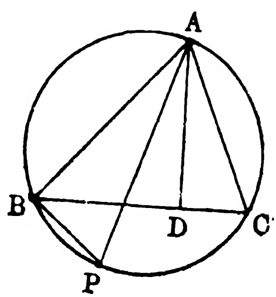
$\therefore$ the $\triangle^s$ BAP and DAC being equiangular are similar.

$\therefore$ $AB : AP = AD : AC$.

$\therefore$ $AB \cdot AC = AP \cdot AD = (AD + DP)AD = AD^2 + AD \cdot DP$
$= AD^2 + BD \cdot DC$.

Ex. 11. *If from the vertical angle of a triangle a perpendicular is drawn to the base, prove that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.* [**Brahmagupta's Theorem**]

Let $ABPC$ be the circum-circle of the $\triangle ABC$, AP the diameter of the circum-circle and AD the perpendicular drawn from A upon the base BC.



It is reqd. to prove that

$$AB \cdot AC = AD \cdot AP.$$

Proof. Join BP.

In the $\triangle^s$ BAP and DAC, the $\angle BPA$ = the $\angle DCA$ (in the same segment) and the $\angle ABP$ = the $\angle ADC$ [$\because$ AD is $\perp$ BC and $\angle ABP$ in a semi-circle = a rt. $\angle$.]

$\therefore$ the $\triangle^s$ BAP and DAC are equiangular, and hence similar.

$\therefore$ $AB : AD = AP : AC$. $\therefore$ $AB \cdot AC = AD \cdot AP$.

N. B. With the help of Brahmagupta's Theorem a useful Trigonometrical formula can be established.

If the sides of triangle ABC opposite to the $\angle A$, the $\angle B$ and the $\angle C$ be denoted by a, b and c respectively, the radius of its circum-circle by R and its area by S, then by the above Theorem it can be easily proved that $R=\frac{abc}{4S}$,

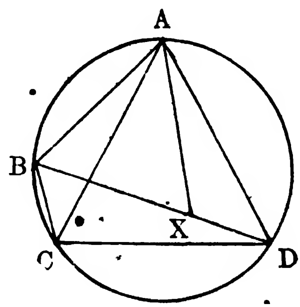
From Brahmagupta's Theorem, we get $AB \,.\, AC = AD \,.\, AP$.

i.e., $c \,.\, b = AD \,.\, 2R$.

$\therefore \quad 2R=\frac{bc}{AD}=\frac{bc \,.\, a}{AD \,.\, a}=\frac{abc}{2S}$ [$\because$ $S=\frac{1}{2}a \,.\, AD$] $\therefore$ $R=\frac{abc}{4S}$.

Ex. 12. *Prove that the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by its two opposite sides.* [**Ptolemy's Theorem**]

Let $ABCD$ be a cyclic quadrilateral and AC, BD its diagonals.



It is reqd. to prove that

$$AC \,.\, BD = AB \,.\, CD + BC \,.\, AD.$$

Construction. Make the $\angle DAX$ equal to the $\angle BAC$ and let AX meet the diagonal BD in X.

Proof. The $\angle BAC$ = the $\angle DAX$. (by construction)

To each of these add the $\angle CAX$.

$\therefore$ the $\angle BAX$ = the $\angle CAD$.

Now, in the $\triangle$' BAX and ACD, the $\angle BAX$ = the $\angle CAD$ (proved) and the $\angle ABX$ = the $\angle ACD$ (in the same segment).

$\therefore$ the $\triangle$' BAX and ACD, being equiangular, are similar.

$\therefore$ $AB : AC = BX : CD$.

$\therefore$ $AB \,.\, CD = AC \,.\, BX$ (by cross-multiplication) ... ... (1)

Again, the $\angle BAC$ = the $\angle DAX$ (by construction) and the $\angle ACB$ = the $\angle ADB$ *i.e.*, $\angle ADX$ (in the same segment).

$\therefore$ the two $\triangle$' ABC and AXD are similar.

$\therefore$ $BC : XD = AC : AD$.

$\therefore$ $BC \,.\, AD = AC \,.\, XD$ (by cross-multiplication) ... (2)

Now by adding (1) and (2), we have

$$AB \,.\, CD + BC \,.\, AD = AC \,.\, BX + AC \,.\, XD$$
$$= AC(BX + XD) = AC \,.\, BD.$$

Exercise II(B)

1. In a triangle ABC the bisector of the vertical angle BAC meets the base at D and the circum-circle at E; show that AB, AD, AE and AC are four proportionals.

2. Prove that the straight line joining the middle points of the parallel sides of a trapezium passes through the point of intersection of the diagonals.

3. Prove that the point of intersection of the oblique sides of a trapezium, the middle points of the parallel sides and the point of intersection of the diagonals are concurrent.

4. Through the point of contact A of two circles which touch each other, a straight line is drawn to cut the two circles at P and Q. Prove that their diameters are as $AP : AQ$.

5. (i) Prove that the median of a triangle bisects all straight lines parallel to the side the median bisects.

(ii) In similar triangles show that the corresponding medians make equal angles with the corresponding sides.

6. Two straight lines AB and CD intersect at O, internally, or externally, so that $OA : OC = OD : OB$. Show that the four points A, C, B, D are concyclic.

7. If two triangles are on equal bases and between the same parallels, prove that the intercepts made by the sides of the triangle on any line parallel to the line of the bases are equal.

8. ABC is a right-angled triangle, and from A, a perpendicular AD is drawn to the hypotenuse BC; prove that

$$BD : DC = AB^2 : AC^2.$$

9. From the angle A of a parallelogram $ABCD$ a straight line is drawn cutting the diagonal BD in P, and the sides BC and DC (produced where necessary) in Q and R respectively; prove that AP is a mean proportional between PQ and PR.

10. (i) In a triangle ABC, the straight line DEF meets the sides BC, CA, AB at the points D, E, F respectively, and makes equal angles with AB and AC; prove that $BD : CD = BF : CE$.

(ii) In a triangle ABC, F and E are points on AB and AC such that $BF = CE$. If FE and BC, when produced, meet at D, prove that $AB : AC = DE : DF$.

11. If two circles touch externally, prove that their common tangent is a mean proportional between their diameters.

12. In two circles any two parallel radii are drawn, one in each circle; prove that the straight line joining their extremities cuts the line of centres in one or other of two fixed points.

13. ABC is a triangle and X is any point in BC; prove that the circum-radii of the triangles ABX and ACX are as $AB : AC$.

14. If the diagonals of a cyclic quadrilateral are at right angles, prove that the sum of the rectangles contained by its opposite sides is equal to twice the area of the quadrilateral.

15. AC and BD are perpendiculars on CD, and AD, BC intersect at P. If PQ be perpendicular to CD, show that PQ bisects the angle AQB.

16. (i) P is the middle point of the median AD of the triangle ABC. If BP produced meet CA at E, prove that $CE = 2AE$.

(ii) P is any point on the median AD of the triangle ABC. If BP and CP, when produced, meet AC and AB at E and F, prove that FE is parallel to BC.

Theorem 8

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing that angle, and likewise the external bisector externally.

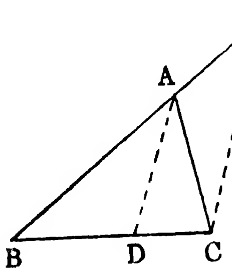


Fig. 1

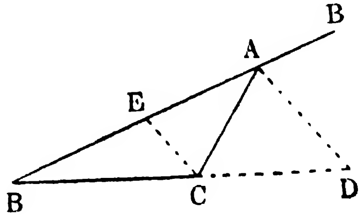


Fig. 2

Let AD, the bisector of the $\angle BAC$ of the $\triangle ABC$ divide the side BC internally in Fig. 1 at the pt. D and in Fig. 2, let the str. line AD bisect the exterior $\angle B'AC$ and cut BC produced at D.

It is reqd. to prove, in both cases, that $BD : CD = AB : AC$.

Construction. Through C draw $CE \parallel$ to DA to meet (i) BA produced at E (Fig. 1) or (ii) BA at E (Fig. 2).

Proof. Since AD is $\parallel$ to EC, $\therefore$ the $\angle DAC =$ the alt. $\angle ACE$ and the $\angle BAD$ or the $\angle B'AD =$ corresponding $\angle AEC$.

But the $\angle BAD$ or $\angle B'AD =$ the $\angle DAC$ (Hyp.).

$\therefore$ the $\angle ACE =$ the $\angle AEC$, $\therefore$ $AC = AE$.

Now, AD is $\parallel$ to EC, $\therefore$ $BD : DC = BA : AE = BA : AC$.

N. B. The converse of this Theorem is also true and this is shown as follows.

Converse Theorem.

If a straight line drawn from an angle of a triangle divides the opposite side (i) internally or (ii) externally in the ratio of the sides containing the angle, the straight line is the (i) internal or (ii) external bisector of that angle.

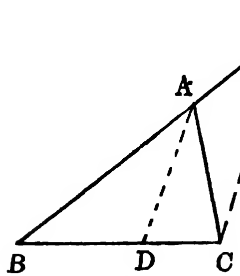


Fig. 1

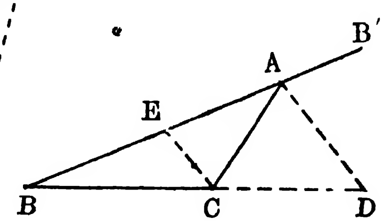


Fig. 2

In the $\triangle ABC$ let the str. line AD drawn from the $\angle A$ divide BC (in Fig. 1) or BC produced (in Fig. 2) at D, so that $BD : DC = BA : AC$.

It is reqd. to prove that AD is the bisector of the $\angle BAC$ (in Fig. 1) or of the external $\angle B'AC$ (in Fig 2).

Construction. Through C draw $CE \parallel$ to DA to meet BA produced at E (in Fig. 1) or BA at E (in Fig. 2).

Proof. Since AD is $\parallel$ to EC, $\therefore$ $BD : DC = BA : AE$.

But $BD : DC = BA : AC$ (Hyp.). $\therefore$ $AE = AC$.

$\therefore$ the $\angle AEC =$ the $\angle ACE$.

Again, since AD and EC are $\parallel$, the $\angle CAD =$ the alt. $\angle ACE$ and in Fig. 1, the $\angle BAD$ or in Fig 2, the $\angle B'AD =$ the corresponding $\angle AEC$.

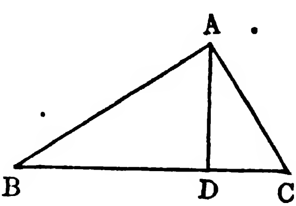
But the $\angle AEC =$ the $\angle ACE$.

$\therefore$ the $\angle BAD$ or $\angle B'AD =$ the $\angle CAD$.

That is, the line AD bisects (i) the $\angle BAC$ (in Fig. 1) or (ii) the ext. $\angle B'AC$ (in Fig. 2).

Theorem 9

In a right-angled triangle if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.



In the $\triangle ABC$ let the $\angle A$ be a rt. $\angle$ and let AD be drawn perpendicular to BC from the angular pt. A.

It is reqd. to prove that the $\triangle^s$ ADB, ADC and ABC are similar.

Proof. In the $\triangle^s$ ADC and ABC, the $\angle ADC =$ the $\angle BAC$ (being rt. $\angle^s$) and the $\angle C$ is common.

$\therefore$ the $\triangle^s$ ADC and ABC are equiangular, and hence similar. Similarly, it may be proved that the $\triangle^s$ ADB and ABC are similar.

$\therefore$ the $\triangle^s$ ADB and ADC, both being equiangular to the $\triangle ABC$, are equiangular to one another.

$\therefore$ the $\triangle^s$ ADC and ADB are similar.

Cor. 1. Δ^s ADC and ADB are similar,

$\therefore$ $CD : AD = AD : BD.$

$\therefore$ **$AD^2 = BD.CD$.**

i.e., AD is a mean proportional between BD and CD.

Cor. 2. Δ^s ADB and ABC are similar,

$\therefore$ $BD : AB = AB : BC.$

$\therefore$ **$AB^2 = BD.BC$.**

i.e., AB is a mean proportional between BD and BC.

Cor. 3. Δ^s ADC and ABC are similar,

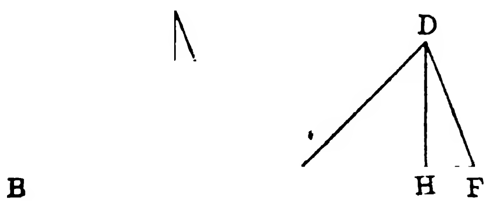
$\therefore$ $CD : AC = AC : BC.$

$\therefore$ **$AC^2 = CD.BC$.**

i.e., AC is a mean proportional between CD and BC.

Theorem 10

The ratio of the areas of similar triangles is equal to the ratio of the squares on their corresponding sides.



Let the two similar Δ^s ABC, DEF have the $\angle A =$ the $\angle D$, the $\angle B =$ the $\angle E$ and the $\angle C =$ the $\angle F$. Therefore, BC and EF are corresponding sides.

It is reqd. to prove that $\Delta ABC : DEF = BC^2 : EF^2$.

Construction. From A and D draw AG and DH perps. to BC and EF respectively.

Proof. In the Δ^s ABG and DEH, the $\angle B =$ the $\angle E$ (Hyp.)

and the $\angle AGB$ = the $\angle DHE$ (being rt. $\angle^s$)

$\therefore$ the Δ^s ABG and DEH are equiangular, and hence similar.

$\therefore$ $AG : DH = AB : DE = BC : EF$

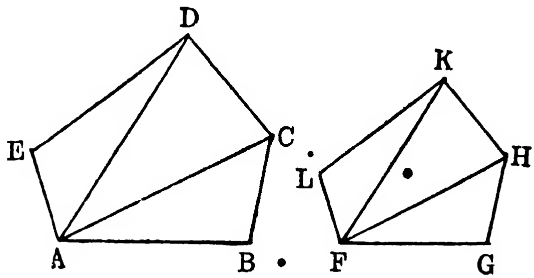
[$\because$ the Δ^s ABC and DEF are similar]

Again, $\Delta ABC = \frac{1}{2} BC \,.\, AG$ and $\Delta DEF = \frac{1}{2} EF \,.\, DH$.

$$\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2} BC \,.\, AG}{\frac{1}{2} EF \,.\, DH} = \frac{BC}{EF} \cdot \frac{AG}{DH} = \frac{BC}{EF} \cdot \frac{BC}{EF} = \frac{BC^2}{EF^2}.$$

[$\because$ the Δ^s are similar.]

Cor. *Areas of two similar polygons are to one another as the squares on their corresponding sides.*



It may be easily proved with the help of Theorem 7 that if two corresponding vertices A, F of two similar polygons $ABCDE$, $FGHKL$ be joined to the other vertices of the respective polygons, the two polygons will be divided into an equal number of similar triangles.

$\therefore$ the Δ^s ABC and FGH, the Δ^s ACD and FHK and the Δ^s ADE and FKL are similar.

$$\therefore \frac{\Delta ABC}{\Delta FGH} = \frac{AC^2}{FH^2} = \frac{\Delta ACD}{\Delta FHK} = \frac{AD^2}{FK^2} = \frac{\Delta ADE}{\Delta FKL}.$$

$$\therefore \frac{\Delta ABC}{\Delta FGH} = \frac{\Delta ACD}{\Delta FHK} = \frac{\Delta ADE}{\Delta FKL}.$$

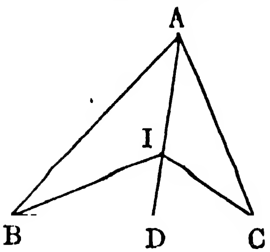
$$\therefore \quad \frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle FGH + \triangle FHK + \triangle FKL} \quad \text{(by Addendo)}$$

$$= \frac{\text{polygon } ABCDE}{\text{polygon } FGHKL}.$$

But $\dfrac{\triangle ABC}{\triangle FGH} = \dfrac{AB^2}{FG^2}$. $\therefore$ $\dfrac{\text{polygon } ABCDE}{\text{polygon } FGHKL} = \dfrac{AB^2}{FG^2}$

2·4. Illustrative Examples.

Ex. 1. *Prove, with the help of Theorem 8, that the internal bisectors of the angles of a triangle are concurrent.*



Let the str. lines BI and CI, the bisectors of the $\angle ABC$ and $\angle ACB$ of the $\triangle ABC$, intersect each other at the pt. I. Join AI.

It is reqd. to prove that the line AI bisects the $\angle BAC$.

Construction. Produce AI to meet BC in D.

Proof. In the $\triangle ABD$, the line BI bisects the $\angle ABD$ (Hyp.)

$\therefore$ $AB : BD = AI : ID$.

Again, in the $\triangle ACD$, the line CI bisects the $\angle ACD$ (Hyp.)

$\therefore$ $AC : CD = AI : ID$.

$\therefore$ $AB : BD = AC : CD$ (each being equal to $AI : ID$)

$\therefore$ $AB : AC = BD : CD$ (by Alternando).

$\therefore$ the line AID bisects the $\angle BAC$. [Theo. 8, Converse]

Hence bisectors of the angles of a triangle are concurrent.

Ex. 2. *Prove that the locus of a point which moves so that the ratio of its distances from two fixed points is constant, is a circle.*

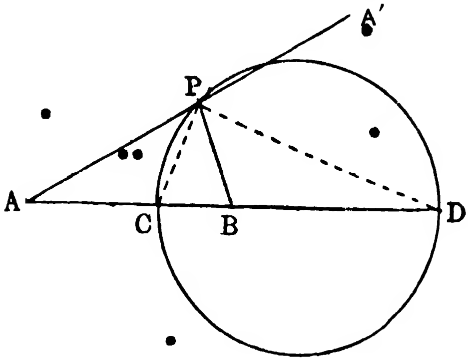
[Circle of Apollonius]

Let A, B be two fixed points and P a moving point such that, for all positions of P, $PA : PB =$ a constant.

It is reqd. to find the locus of P.

Construction. Let P be one position of the moving point. Join AP, BP and produce AP to A'. Let the lines PC, PD bisect the $\angle$s APB, $A'PB$ respectively and let them meet AB and AB produced at the points C, D.

The circle described on CD as diameter will be the reqd. locus.



Proof. Since PC bisects the $\angle APB$;

$\therefore$ $AC : BC = PA : PB =$ a constant.

$\therefore$ C is a fixed point. [$\because$ A, B are fixed pts.]

Similarly, it may be proved that D also is a fixed pt.

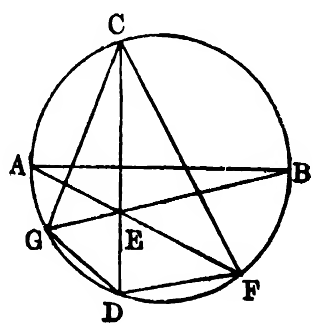
$\therefore$ the position of CD is fixed.

Again, since PC, PD are the bisectors of the adjacent $\angle$s APB, $A'PB$, the $\angle CPD =$ a rt. $\angle$.

$\therefore$ the fixed str. line CD always subtends a rt. $\angle$ at P.

N. B. This circle is called the **Circle of Apollonius.**

Ex. 3. *AB is a diameter of a circle, CD is a chord at right angles to it and E any point in CD ; AE and BE are drawn and produced to cut the circle in F and G ; show that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.*



Let AB be a diameter of the circle and CD a chord at right angles to AB.

Let E be a pt. in CD and let AE, BE be produced to cut the circle at F, G respectively.

It is reqd. to prove that the quadrilateral $CFDG$ has any two of its adjacent sides in the same ratio as the remaining two.

Proof. Since the diameter AB is perp. to chord CD, AB bisects CD at rt. $\angle^s$.

$\therefore$ the arc BC = the arc BD and the arc AC = the arc AD.

$\therefore$ the $\angle BGC$ = the $\angle BGD$ and the $\angle AFC$ = the $\angle AFD$

(Standing on equal arcs)

$\therefore$ the lines GE and FE bisect the $\angle^s$ CGD and CFD respectively.

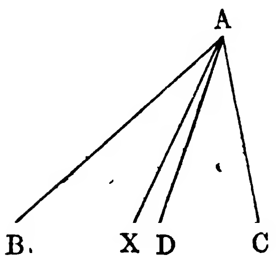
$\therefore$ $CG : GD = CE : ED$ and $CF : FD = CE : ED$.

$\therefore$ $CG : GD = CF : FD$.

$\therefore$ $CG : CF = GD : FD$ (by Alternando).

i.e. any two sides of the quadrilateral $CFDG$ have the same ratio as the remaining two.

Ex. 4. *The angle A of a triangle ABC is bisected by AD meeting BC in D and AX is the median bisecting BC; show that XD has the same ratio to XB as the difference of the sides has to their sum.*



Let the bisector AD of the $\angle A$ of the $\triangle ABC$ meet the side BC at D and let AX the median bisect the side BC at X.

It is reqd. to prove that $XD : XB = (AB - AC) : (AB + AC)$.

Proof. Since AD bisects the $\angle BAC$,

$$\frac{AB}{AC} = \frac{BD}{CD}.$$

$$\therefore \frac{AB+AC}{AB-AC} = \frac{BD+CD}{BD-CD} \quad \text{(by Componendo-Dividendo)}$$

$$= \frac{BC}{(BX+XD)-(CX-XD)} = \frac{2XB}{2XD} \quad [\because BX = CX]$$

$$= \frac{XB}{XD}.$$

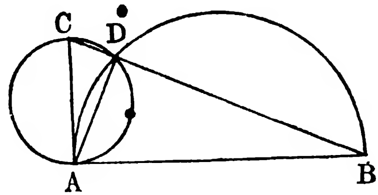
$$\therefore \frac{XD}{XB} = \frac{AB-AC}{AB+AC} \quad \text{[by Invertendo]}$$

i.e. $XD : XB = (AB - AC) : (AB + AC)$.

Ex. 5. *In a right-angled triangle, one side about the right angle is double the other; if circles are described on these sides as diameters, prove that their common chord is two-fifths of the hypotenuse.*

In the rt.-$\angle^d$ $\triangle ABC$ let the $\angle A$ be a rt. $\angle$ and let $AB = 2AC$. Let AL

be the common chord of two circles ADC and ADB described on AC and AB as diameters.



It is reqd. to prove that $AD=\frac{2}{5}$ of the hypotenuse BC.

Proof. Suppose the circle described on AC as diameter cuts the hypotenuse at D.

$\therefore$ the $\angle ADC$ being in a semicircle = a rt. $\angle$.

$\therefore$ the $\angle ADB$ being adjacent to it = a rt. $\angle$.

$\therefore$ the circle described on AB as diamater passes through D. Now, in the rt.-$\angle^d$ $\triangle ABC$, AD is perpendicular to the hypotenuse BC.

$\therefore$ $BC \,.\, BD=AB^2$ and $BC \,.\, CD=AC^2$. (Cor., Theor. 9)

$$\frac{BC \,.\, BD}{BC \,.\, CD}=\frac{AB^2}{AC^2}=4. \quad [\because AB=2AC]$$

$$i.e. \quad \frac{BD}{CD}=4. \qquad BD=4CD.$$

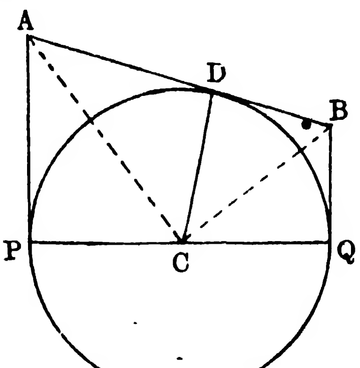
Now $BC=BD+CD=4CD+CD=5CD$.

$\therefore$ $CD=\frac{1}{5}BC$. Hence $BD=\frac{4}{5}BC$.

Again, $AD^2=BD \,.\, CD=\frac{4}{5}BC \,.\, \frac{1}{5}BC=\frac{4}{25}BC^2$.

$\therefore$ $AD=\frac{2}{5}BC$.

Ex. 6. *Any tangent to a circle is drawn to cut a pair of parallel tangents; prove that the radius of the circle is a mean proportional between the segments into which the tangent is divided at the point of contact.*



Let PA and QB be two parallel tangents at the pts. P and Q on the circumference of a circle whose centre is O and let another tangent to the circle at the pt. D on it cut the two former tangents at A and B. Join CD.

It is reqd. to prove that

$$CD^2=AD \,.\, DB.$$

Proof. Join CP, CQ, CA, CB.

Since PA, PB are tangents and CP, CQ are radii, each of the $\angle^s$ CPA, CQB is a rt. $\angle$ and PA, QB are parallel.

$\therefore$ P, C, Q are collinear.

[It can be easily proved by drawing through C a line $\parallel$ to PA].

Now in the $\triangle$' APC, ADC, the $\angle ADC$ = the $\angle APC$ [each being the $\angle$ made by a radius with the tangent is a rt.-$\angle$], the hypotenuse AC is common and $CP = CD$ (being radii)

$\therefore$ the two triangles are equal in all respects.

$\therefore$ the $\angle ACD$ = the $\angle ACP$, *i.e.* the $\angle ACD = \frac{1}{2}\angle PCD$.

Similarly, it can be proved that the $\angle BCD = \frac{1}{2}\angle QCD$.

$\therefore$ the $\angle ACD$ + the $\angle BCD = \frac{1}{2}$(the $\angle PCD$ + the $\angle QCD$) = a rt. $\angle$

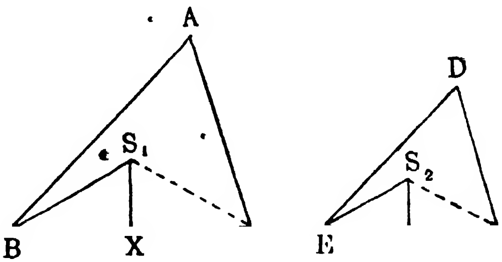
i.e. the $\angle ACB$ = a rt. $\angle$.

Now, the $\triangle ABC$ is a rt.-$\angle^d$ $\triangle$ and CD is perp. to the hypotenuse AB,

[$\because$ AB is a tangent and CD a radius]

$\therefore$ $CD^2 = AD . DB$.

Ex. 7. *Prove that the areas of two similar triangles are to one another as the squares on the radii of their circum-circles.*



Let the $\triangle$' ABC, DEF be similar and S_1, S_2 their circum-centres. Hence S_1B, S_2E are their circum-radii.

It is reqd. to prove that the $\triangle ABC : \triangle DEF = S_1B^2 : S_2E^2$.

Construction. From the pts. S_1, S_2 draw the perps. S_1X, S_2Y to the sides BC, EF respectively. Join S_1C, S_2F.

Proof. S_1X, S_2Y are perps. to the chords BC, EF respectively from the pts. S_1, S_2 the centres of the two circum-cricles.

$\therefore$ $BX = XC$ and $EY = YF$.

Now, in the $\triangle$' BS_1X, CS_1X, $S_1B = S_1C$ (being radii of the circum circle), $BX = XC$ and S_1X is common.

$\therefore$ the two $\triangle$' are equal in all respects.

$\therefore$ the $\angle BS_1X$ = the $\angle CS_1X = \frac{1}{2}\angle BS_1C$.

Again, S_1 is the centre of the circum-circle and A a pt. on its circumference,

$\therefore$ the $\angle BS_1C = 2\angle BAC$. $\therefore$ the $\angle BS_1X =$ the $\angle BAC$.

Similarly, the $\angle ES_2Y =$ the $\angle EDF$.

But since, the $\triangle^s$ ABC, DEF are similar, the $\angle BAC =$ the $\angle EDF$.

$\therefore$ the $\angle BS_1X =$ the $\angle ES_2Y$.

Hence, in the $\triangle^s$ BS_1X, ES_2Y,

the $\angle BS_1X =$ the $\angle ES_2Y$ (proved) and the $\angle S_1XB =$ the $\angle S_2YE$ (rt. $\angle^s$)

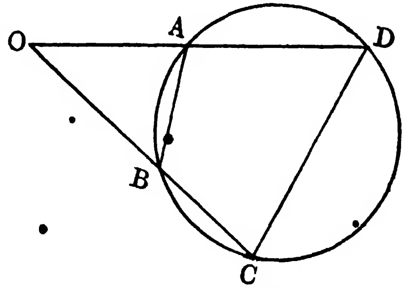
$\therefore$ the $\triangle^s$ BS_1X, ES_2Y are equiangular, and hence similar.

$\therefore$ $S_1B : S_2E = BX : EY = \frac{1}{2}BC : \frac{1}{2}EF = BC : EF$.

Now, since the $\triangle^s$ ABC, DEF are similar,

$$\triangle ABC : \triangle DEF = BC^2 : EF^2$$
$$= S_1B^2 : S_2E^2.$$

Ex. 8. *ABCD is a cyclic quadrilateral of which the sides DA, CB are produced to meet in O. If AB is half of CD, prove that the quadrilateral ABCD is three times the triangle OAB.*



Let $ABCD$ be a cyclic quadrilateral in which $CD = 2AB$ and let DA, CB be produced to meet in O.

It is reqd. to prove that the quadrilateral $ABCD = 3\triangle OAB$.

Proof. The exterior $\angle OAB$ of the cyclic quadrilateral $ABCD$ = the opposite interior $\angle BCD$.

Now, in the $\triangle^s$ OAB, OCD, the $\angle OAB =$ the $\angle BCD$ (proved) *i.e.* the $\angle OCD$ and the $\angle AOB =$ the $\angle COD$ (the same angle).

$\therefore$ the $\triangle^s$ OAB, OCD being equiangular are similar.

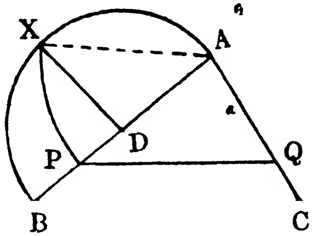
$$\therefore \quad \triangle OCD : \triangle OAB = CD^2 : AB^2 = 4 : 1.$$
$$\therefore \quad \triangle OCD = 4\triangle OAB.$$

Again the quadrilateral $ABCD = \triangle OCD - \triangle OAB$
$$= 4\triangle OAB - \triangle OAB$$
$$= 3\triangle OAB.$$

Ex. 9. ***Show how to bisect a triangle by a straight line drawn parallel to the base.***

Let ABC be the given $\triangle$.

It is reqd. to bisect the $\triangle ABC$ by a straight line drawn parallel to the base BC.



Construction. Bisect AB at D. With centre D and radius AD describe a semi-circle on AB. At the pt. D in AB draw DX perp. to AB. Let DX cut the semi-circle at the pt. X. Join AX. From AB cut off $AP = AX$.

Through P draw $PQ \parallel$ to BC to meet AC in Q.

Then PQ shall bisect the $\triangle ABC$.

Proof. Now, $AP^2 = AX^2 = AD^2 + DX^2 = 2AD^2$ [$\because AD = DX$, being radii]
$= \frac{1}{2}AB^2$. [$\because AB = 2AD$]
$\therefore AB^2 = 2AP^2$.

Now, since PQ is $\parallel$ to BC, the $\triangle$s APQ, ABC being equiangular are similar.

$\therefore \triangle APQ : \triangle ABC = AP^2 : AB^2 = AP^2 : 2AP^2 = 1 : 2$.
$\therefore$ the line $\triangle APQ = \frac{1}{2}\triangle ABC$,
i.e. the line PQ bisects the $\triangle ABC$.

Exercise II(C)

1. If the bisector of the vertical angle of a triangle also bisects the base, prove that the triangle is isosceles.

2. The angle BAC of a triangle ABC is bisected by AD meeting BC in D and the angle ABC is bisected by BI meeting AD in I; prove that $AI : ID = (AB + AC) : BC$.

3. The angle BAC of a triangle ABC is bisected by AD meeting BC in D and O is the middle point of BC; if $AB > AC$, prove that $OB : OD = (AB + AC) : (AB - AC)$.

4. From any point P on a circle described on AB as diameter, PC and PD are drawn making equal angles with AP and meeting AB in C and D; prove that $AD : DB = AC : CB$.

5. AD is the median of a triangle ABC and the angles ADB and ADC are bisected by DE and DF meeting AB and AC at E and F respectively. Prove that EF is parallel to BC.

6. If the bisectors of one pair of opposite angles of a quadrilateral meet on one of the diagonals, prove that the bisectors of the other pair of opposite angles will meet on the other diagonal.

7. Prove that in any triangle the bisectors of two exterior angles and the bisector of the third interior angle are concurrent.

8. A straight line AB is divided internally at C and externally at D in the same ratio, and on CD as diameter a circle is described. If P be any point on this circle, show that PC and PD are the internal and external bisectors respectively of the angle APB.

9. AB is a fixed straight line and C a fixed point in it and through C any straight line CD is drawn ; find a point P in CD such that $AP : PB = AC : CB$.

10. The internal and external bisectors of the vertical angle APB cut the base AB in C and D respectively ; if O is the middle point of AB, prove that $OB^2 = OC \cdot OD$.

11. ABC is a triangle right-angled at A and AB is greater than AC ; AD is drawn perpendicular to BC. If BC, AB and AC are in continued proportion, show that $AC = BD$.

12. AD and BE are the medians of the triangle ABC which meet at G ; if DE be joined, compare the areas of the triangles AGB and DGE.

13. Prove that the areas of two similar triangles are to one another as the squares on

(i) their corresponding altitudes ;

(ii) their corresponding medians ;

and (iii) the radii of their in-circles.

14. DEF is the pedal triangle of the triangle ABC. Prove that $\triangle ABC : \triangle CDE = BC^2 : CE^2$.

15. In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, show that the segments of the hypotenuse have the same ratio as the squares on the sides containing the right angle.

16. DE is drawn parallel to the base BC of a triangle ABC meeting AB and AC in D and E so that the triangle ADE is one-ninth of the whole triangle ; find the ratio of AD to DB.

17. If ABC be a triangle and D a point in BC such that BC is equal to the diagonal of the square on DC, show that a line through D parallel to AB bisects the triangle.

18. ABC is a triangle whose area is 16 sq. cm., and XY is drawn parallel to BC, dividing AB in the ratio 3 : 5 ; if BY is joined, find the area of the triangle BXY.

19. In a rectangle $ABCD$, square on AB is double that on AD. If perpendiculars be drawn from A and C on the diagonal BD, show that they will divide BD into three equal parts.

20. Show that the area of a regular hexagon describrd about a circle. is to the area of a regular hexagon inscribed in it as 4 to 3.

Higher Secondary
Elective Mathematics : Paper II

•PLANE GEOMETRY

(Syllabus for Class X)

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle. [*Theorem 11 Page 53*]

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other. [*Theorem 12 Page 55*]

[*Note*—This proposition may be proved with the help of the properties of similar triangles.]

Practical

Construction of tangents to a circle and of common tangents to two circles (both cases). Construction of regular figures of 3, 4, 5 or 6 sides in or about a circle. [*Problems 1, 2, 3, 4, 8, 9, 10, 11 Pages 67-82*]

Construction of a mean proportional to two given straight lines. [*Problem 15 Page 87*]

Construction of a square equal in area to a given polygon. [*Problem 13 Page 85*]

[*Chapters III—IV, Pages 51 to 100*]

CONTENTS

PLANE GEOMETRY

(Course for Class X)

CHAPTER III

SECANT AND TANGENT TO A CIRCLE

3·1. Secant and Tangent to a circle.

A straight line generally cuts a circle in two points. It is then called *secant* to the circle. When the two points of intersection of the secant with the circle coincide with one another, the secant is called a *tangent* to the circle. In Fig. 1 below the straight line PQ is a secant to the circle. In fig. 2 if the secant PQ be continuously turned towards the left about the point P in it, then the second point of intersection Q will gradually move towards, and finally coincide and become one with, the point P. Then in this limiting position the secant PQ becomes the tangent PT.

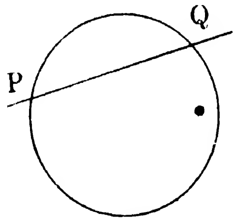


Fig. 1

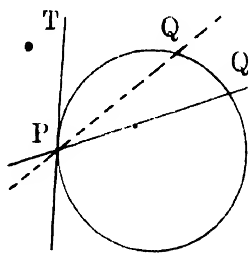


Fig. 2

Hence the straight line which touches a circle at one point only and does not meet it at any other point, even when produced, is called a tangent to the circle. The point on the circle at which the tangent touches it is called the *point of contact.*

Contact of two circles.

Just as a straight line cuts a circle in two points, in a like manner two circles cut one another also in two points. If the two points of intersection of the straight line with the circle coincide with one another, the straight line touches the circle. In a similar way, if the two points of intersection of the two circles coincide with one another, the two circles are said to touch one another. The contact of two circles may be in two ways, internally or externally.

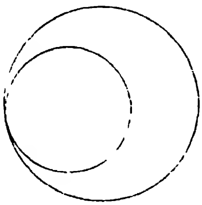


Fig. 1

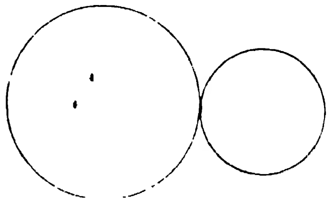


Fig. 2

When one circle lies entirely within the other and touches it (Fig. 1), the contact is said to be *Internal Contact* ; and when one circle lies entirely outside the other and touches it (Fig. 2), the contact is said to be *External Contact.*

Common Tangent to two circles.

When a straight line simultaneously touches two circles, then it is called a Common Tangent to the two circles. A common tangent may be of two kinds—direct and transverse. If the two points of contact of the common tangent to the two circles be on the same side of the line joining the centres of the two circles, the tangent is said to be a **Direct Common Tangent,** and if the two points of contact lie on both sides of the line joining the centres of the two circles, the tangent is said to be a **Transverse Common Tangent.**

Of two circles when one lies entirely within the other without touching it (Fig. 1), they have no common tangent. If the two circles touch internally (Fig. 2), they have only one common tangent at the point of contact. If the two circles touch

externally (Fig. 3), they have three common tangents of which one is at the point of contact and the remaining two are direct

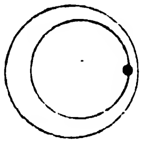


Fig. 1

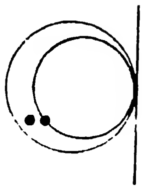


Fig. 2

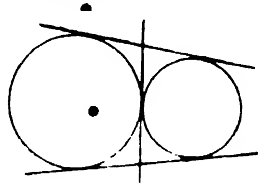


Fig. 3

common tangents. If the two circles cut one another (Fig. 4), they have two direct common tangents only and if without

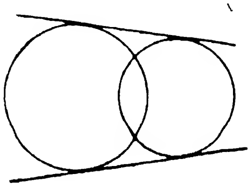


Fig. 4

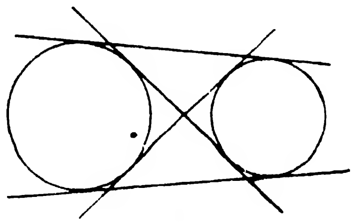
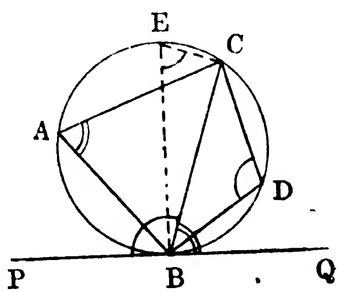


Fig. 5

cutting, one of the circles lies wholly outside the other, there are four common tangents—two direct and the other two transverse.

Theorem 11

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let the straight line PQ touch the circle $ABDC$ at the pt. B and BC be the chord of contact drawn from the pt. B.

It is reqd. to prove that

the $\angle CBQ$ = the $\angle BAC$ in the alternate segment,

and the $\angle CBP$ = the $\angle BDC$ in the alternate segment.

Construction. Draw the diameter through the pt. B and join EC.

Proof. Since the $\angle ECB$ is a rt. $\angle$ (being in a semi-circle)

$\therefore$ the $\angle BEC +$ the $\angle EBC =$ a rt. $\angle$.

Again, PBQ is a tangent and BE a diameter through the pt. of contact.

$\therefore$ the $\angle EBQ =$ a rt. $\angle =$ the $\angle EBC +$ the $\angle CBQ$.

$\therefore$ the $\angle BEC +$ the $\cancel{\angle EBC} =$ the $\cancel{\angle EBC} +$ the $\angle CBQ$.

Removing the common $\angle EBC$ from both sides,

the $\angle CBQ =$ the $\angle BEC$.

But the $\angle BEC =$ the $\angle BAC$ in the same segment.

$\therefore$ the $\angle CBQ =$ the $\angle BAC$; which is in the alternate segment.

Again, the $\angle CBP$ is the supplement of the $\angle CBQ$.

$\therefore$ the $\angle CBP$ is the supplement of the $\angle BAC$.

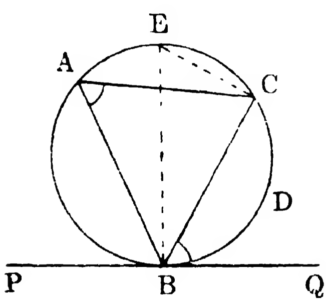
But $ABDC$ is a cyclic quadrilateral.

$\therefore$ the $\angle BAC$ is the supplement of the $\angle BDC$.

$\therefore$ the $\angle CBP =$ the $\angle BDC$, which is in the alternate segment.

Converse Theorem

If through an extremity of a chord of a circle a straight line is drawn making with the chord an angle equal to the angle in the alternate segment, then the straight line so drawn touches the circle.



Let BC be a chord of the circle $ABDC$ and let the $\angle CBQ$ made by the line PBQ drawn through the extremity B of the chord with BC be equal to the $\angle BAC$ in the alternate segment.

It is reqd. to prove that the str. line PBQ is a tangent to the circle.

Construction. Draw the diameter of the circle through B and join EC.

Proof. Now the $\angle BEC$ = the $\angle BAC$ in the same segment
= the $\angle CBQ$ (Hyp.).

Again, the $\angle ECB$ = a rt. $\angle$, being in a semi-circle.

$\therefore$ the $\angle EBC$ + the $\angle BEC$ = a rt. $\angle$.

$\therefore$ the $\angle EBC$ + the $\angle CBQ$ = a rt. $\angle$

[$\because$ the $\angle BEC$ = the $\angle CBQ$, proved]

i.e. the $\angle EBQ$ = a rt. $\angle$.

But the line BE is a diameter,

$\therefore$ PBQ is a tangent, because a tangent is at rt. $\angle^s$ to the radius drawn through the pt. of contact.

Theorem 12

If two chords of a circle intersect, either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

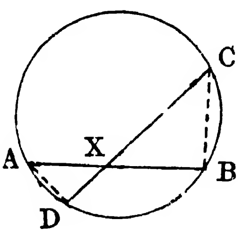


Fig. 1

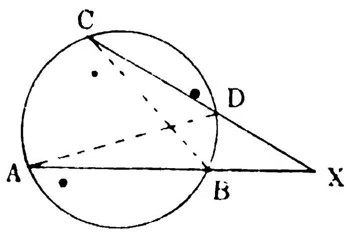


Fig. 2

Let the two chords AB, CD of the circle ABC intersect, inside the circle in Fig. 1 and outside the circle in Fig. 2, at the point X.

It is reqd. to prove in both figures that $AX \cdot XB = CX \cdot XD$.

Construction. Join AD, BC.

Proof. In the Δ^s AXD, BXC, the $\angle AXD$ = the $\angle BXC$ [vertically opp. $\angle^s$ in Fig. 1 and the same $\angle$ in Fig. 2]

and the $\angle XAD$ = the $\angle XCB$ in the same segment.

$\therefore$ the Δ^s AXD, BXC being equiangular, are similar.

$\therefore$ $AX : CX = XD : XB$.

$\therefore$ $AX . XB = CX . XD$. (by cross multiplication).

Alternative Proof.

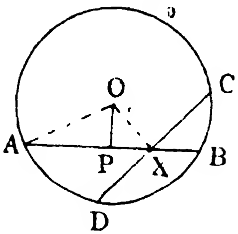


Fig. 1

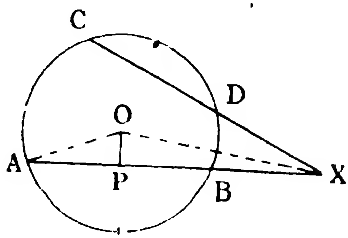


Fig. 2

Construction. From the centre O of the circle draw OP perp. to the chord AB.

Join OA, OX.

Proof. Since OP is perp. to the chord AB, $\therefore$ $AP = PB$.

Now, in Fig. 1, $AX . XB = (AP + PX)(PB - PX)$

$= (AP + PX)(AP - PX)$ [$\because AP = PB$]

$= AP^2 - PX^2$

$= (OA^2 - OP^2) - (OX^2 - OP^2)$

$= OA^2 - OX^2$

$= (\text{radius})^2 - OX^2$.

In Fig. 2, $AX . XB = (PX + AP)(PX - PB)$

$= (PX + AP)(PX - AP)$ [$\because AP = PB$]

$= PX^2 - AP^2$

$= (OX^2 - OP^2) - (OA^2 - OP^2)$

$= OX^2 - OA^2$

$= OX^2 - (\text{radius})^2$.

Similarly, it may be proved that $CX . XD$

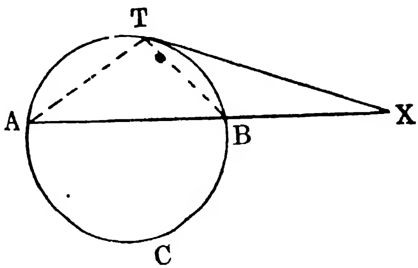
$= (\text{radius})^2 - OX^2$ in Fig. 1

or, $= OX^2 - (\text{radius})^2$ in Fig. 2

$\therefore$ $AX . XB = CX . XD$.

Cor. *If a chord of a circle passes through a point outside the circle, the rectangle contained by its parts is equal to the square on the tangent to the circle from the point.*

Let the chord AB of the circle ACB be produced to the pt. X outside the circle and XT a tangent to the circle.



It is reqd. to prove that $AX.XB = XT^2$.

Construction. Join AT BT.

Proof. In the Δ^s ATX, BTX,

the $\angle BTX$ = the $\angle BAT$, *i.e.* the $\angle XAT$ in the alternate segment. [$\because$ TX is a tangent]

and the $\angle BXT$ = the $\angle AXT$ [the same angle]

$\therefore$ the Δ^s ATX, BTX being equiangular are similar.

$\therefore$ $AX : XT = XT : XB$.

$\therefore$ by cross-multiplication, $XT^2 = AX \,.\, XB$.

N. B. An independent proof of this corollary is given here as it is of great importance. In fact it can be easily proved from Theorem 12. If the two points of intersection C, D of the secant CDX gradually approach one another and ultimately coincide with the pt. T, the secant CDX becomes the tangent XT and $CX = DX = XT$. Hence $CX \,.\, XD = XT^2$.

Converse Theorem

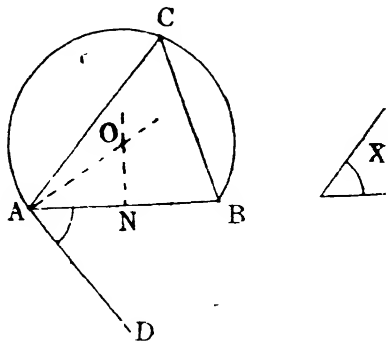
If two straight lines intersect internally or externally, so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, then the extremities of the lines are concyclic.

This Converse Theorem can be easily proved, the proof whereof is left to the students as an exercise.

The converse of the above corollary is also true. Its proof is also left to students.

3·2. Illustrative Examples.

Ex. 1. *On a given finite straight line show how to describe a segment of a circle which shall contain an angle equal to a given angle.*



Let AB be the given straight line and X the given angle.

It is reqd. to describe on AB a segment of a circle which shall contain an angle equal to the given $\angle X$.

Construction. At the pt. A in AB make the $\angle BAD$ equal to the $\angle X$ and draw AO perp. to AD. From the middle pt. N of AB draw a line perp. to AB to meet AO at the pt. O. With centre O and radius equal to OA or OB describe an arc of a circle on the side of AB remote from D. Then this segment of the circle will be the required segment.

Proof. AD is a tangent to the circle at the pt. A, since it is at rt. $\angle^s$ to the radius OA.

$\therefore$ the $\angle BAD$ = the $\angle ACB$ in the alternate segment.

i.e. the $\angle ACB$ = the given $\angle X$.

Ex. 2. *From an external point P tangents are drawn to a circle of centre O and OP cuts the chord of contact at Q; show that (i) $OP.PQ=(tangents)^2$ and (ii) $OP.OQ=(radius)^2$.*

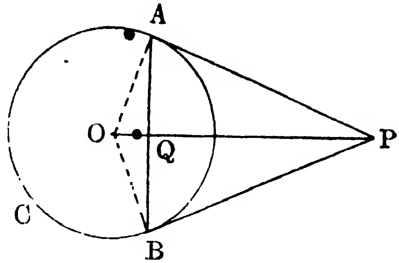
Let PA and PB be two tangents drawn from an external point P to the circle ABC whose centre is O and let AB be the chord of contact. Join OP and let it cut AB in Q.

It is reqd. to prove that (i) $OP.PQ=PA^2$

and (ii) $OP.OQ=OA^2$.

Construction. Join OA, OB.

Proof. Now in the $\triangle^s$ OAP, OBP,



$OA=OB$ (being radii), $PA=PB$ [tangents to the circle from P] and OP is common to both.

$\therefore$ the $\triangle^s$ OAP, OBP are equal in all respects.

$\therefore$ the $\angle OPA=$ the $\angle OPB$.

Again, in the $\triangle^s$ PAQ, PBQ,

$PA=PB$. QP is common to both and the included $\angle QPA$ = the included $\angle QPB$.

$\therefore$ $AQ=QB$ and the $\angle AQP=$ the $\angle BQP=$ a rt. $\angle$. (adjacent $\angle^s$)

Since, OA, OB are radii and PA, PB tangents,

$\therefore$ each of the $\angle^s$ OAP, OBP is a rt. $\angle$,

$\therefore$ the four points O, A, P, B are concyclic.

$\therefore$ $OQ\,.\,QP=AQ\,.\,QB=AQ^2$.

Now (i) $OP\,.\,PQ=(OQ+QP)\,.\,PQ=OQ\,.\,PQ+PQ^2$

$=AQ^2+PQ^2=PA^2$ [$\angle AQP=$ a rt. $\angle$]

(ii) $OP\,.\,OQ=(OQ+QP)\,.\,OQ=OQ^2+OQ\,.\,QP$.

$=OQ^2+AQ^2=OA^2$. [$\angle AQO=$ a rt. $\angle$]

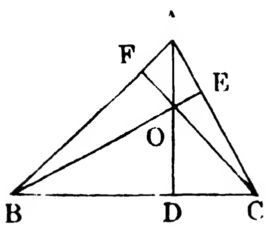
Alternative Proof.

In the $\triangle OAP$, the $\angle OAP=$ a rt. $\angle$ and AQ is perp. to the hypotenuse OP.

$\therefore$ by Cor. 2 and Cor. 3 of Theorem 9

$OP\,.\,OQ=OA^2$ and $OP\,.\,PQ=PA^2$.

Ex. 3. *AD, BE and CF are the perpendiculars drawn from the vertices of a triangle ABC to the opposite sides and O is the orthocentre ; show that $AO\,.\,OD=BO\,.\,OE=CO\,.\,OF$ and that $AD\,.\,AO=AB\,.\,AF=AC\,.\,AE$.*



Let AD, BE, CF be perps. to the sides BC, CA, AB respectively of the $\triangle ABC$ and O its orthocentre.

It is reqd. to prove that

(i) $AO\,.\,OD=BO\,.\,OE=CO\,.\,OF$,

and (ii) $AD\,.\,AO=AB\,.\,AF=AC\,.\,AE$.

Proof. Since each of the $\angle^s$ ADB, AEB is a rt. $\angle$,

$\therefore$ the four points, A, B, D, E are concyclic and the chords AD, BE of this circle intersect at O.

$\therefore \quad AO \cdot OD = BO \cdot OE.$

Similarly, it may be proved that $BO \cdot OE = CO \cdot OF$.

$\therefore \quad AO \cdot OD = BO \cdot OE = CO \cdot OF.$

Again, each of the $\angle^s$ BFO, BDO is a rt. $\angle$,

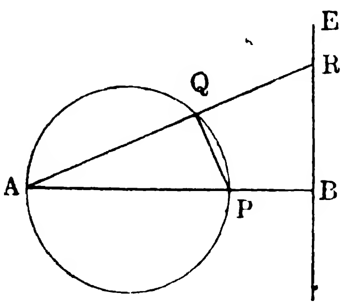
$\therefore$ the four points B, D, O, F are concyclic and the two chords OD, BF of this circle intersect at the pt. A outside the circle.

$\therefore \quad AB \cdot AF = AD \cdot AO.$

Similarly, it may be proved that $AD \cdot AO = AC \cdot AE$.

$\therefore \quad AB \cdot AF = AD \cdot AO = AC \cdot AE.$

Ex. 4. *P is any point in a fixed straight line AB, and AQ is any chord of the circle described on AP as diameter ; R is a point in AQ produced such that AQ . AR = AP . AB. Find the locus of R.*



Let AB be the fixed straight line, P any pt. on it and AQ a chord of the circle described on AP as diameter. Let R be a pt. in AQ, produced such that $AQ \cdot AR = AP \cdot AB$.

It is reqd. to find the locus of R.

Proof. Join BR.

Now, $\because AP \cdot AB = AQ \cdot AR$

$\therefore$ the four points P, Q, R, B are concyclic [Theorem 12, Converse]

$\therefore$ the $\angle PQR$ + the $\angle PBR$ $= 2$ rt. $\angle^s$.

But the $\angle AQP$ is a rt. $\angle$, being in a semi-circle.

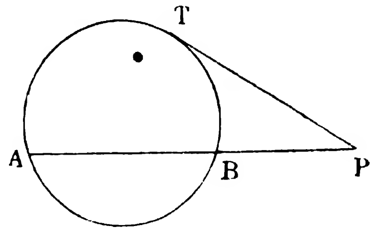
$\therefore$ the adjacent $\angle PQR$ is also a rt. $\angle$.

$\therefore$ the $\angle PBR$ is a rt. $\angle$.

$\therefore$ for all positions of the pt. R, it must be on the perp. to AB through B.

$\therefore$ this str. line must be the locus of R.

Ex. 5. *Circles are drawn through two fixed points A and B, and from a fixed point in AB produced tangents are drawn to them ; prove that the locus of the points of contact of these circles is a circle.*



Let A, B, be two fixed points and P any point in AB produced ; let PT be a tangent to any circle passing through A and B.

It is reqd. to show that the locus of T is a circle.

Proof. Now $AP \cdot PB = PT^2$
[Cor. Theorem 12]

Again, since the three pts. A, B, P are fixed,

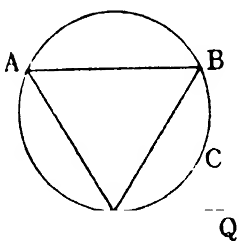
$\therefore$ $AP \cdot PB =$ a constant.

$\therefore$ $PT^2 =$ a constant i.e. PT is constant.

Hence, the length of the tangent drawn to any circle passing through A and B from the fixed pt. P is constant.

$\therefore$ the locus of P is a circle with centre at P.

Ex. 6. *If a tangent to a circle is parallel to a chord, the point of contact is the middle point of the arc cut off by the chord.*



Let the tangent PTQ at the pt. T of the circle ABC be parallel to the chord AB.

It is reqd. to prove that the pt. of contact T is the middle pt. of the arc ACB i,e. the arc $AT =$ the arc BT.

Construction. Join AT, BT.

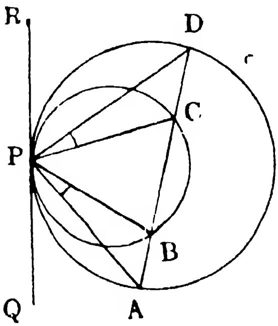
Proof. Since PT is || to AB,

the $\angle ABT =$ the alternate $\angle BTQ$

$=$ the $\angle BAT$ in the alternate segment.

the arc $AT =$ the arc BT.

Ex. 7. *Two circles touch internally and a straight line is drawn to cut them; prove that the parts of it intercepted between the circles subtend equal angles at the point of contact.*



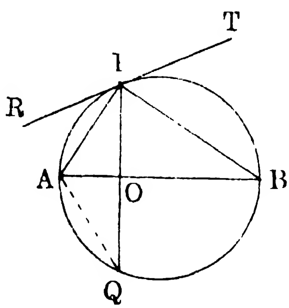
Let two circles touch internally at the pt. P and let the line AD cut the two circles at the pt. A, B, C, D respectively.

It is reqd. to prove that the $\angle APB$ = the $\angle CPD$.

Construction. At P, the pt. of contact of the two circles, draw the common tangent QPR.

Proof. The $\angle APB$
= the $\angle BPQ$ − the $\angle APQ$
= the $\angle BCP$ − the $\angle ADP$ *i.e.* the $\angle CDP$
(in the alternate segment)
= the $\angle CPD$.

Ex. 8. *AB is the diameter of a circle and PQ is a chord at right angles to it; prove that AP and BP are the bisectors of the angles formed by PQ with the tangent at P.*



Let AB be the diameter of the circle APB and PQ a chord of the circle perp. to the diameter AB, let RPT be the tangent to the circle at the pt. P.

It is reqd. to prove that AP and BP bisect the $\angle^s$ RPQ and TPQ respectively.

Construction. Join AQ and let O be the pt. of intersection of AB and PQ.

Proof. Since the diameter AB is perp. to PQ, ∴ O is middle pt. of PQ.

∴ the arc AP = the arc AQ,

∴ the $\angle AQP$ = the $\angle APQ$.

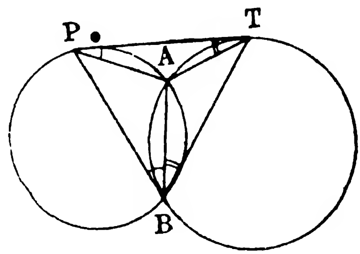
Again, the $\angle APR$ = the $\angle AQP$ in the alternate segment.

= the $\angle APQ$. (proved)

$\therefore$ AP bisects the $\angle BPQ$.

Similarly, by joining BQ it may be proved that BP bisects the $\angle TPQ$.

Ex. 9. *PT is a common tangent to two circles which intersect at A and B; prove that the angles PAT and BAT are supplementary.*



Let the two circles intersect each other at the pts. A, B and let PT be their common tangent.

It is reqd. to prove that the

$\angle PAT +$ the $\angle PBT = 2$ rt. $\angle$.

Construction. Join AP, AT, BP, BT and AB.

Proof. Since PT is a tangent to the circle and PA the chord of contact, the $\angle APT = \angle ABP$ in the alternate segment.

Similarly, the $\angle ATP =$ the $\angle ABT$ in the alternate segment.

$\therefore$ the $\angle ABP +$ the $\angle ABT =$ the $\angle APT +$ the $\angle ATP$.

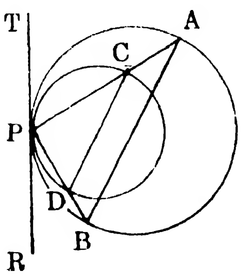
i.e. the $\angle PBT =$ the $\angle APT +$ the $\angle ATP$.

To each of these equals add the $\angle PAT$,

$\therefore$ the $\angle PBT +$ the $\angle PAT =$ the $\angle APT +$ the $\angle ATP +$ the $\angle PAT$

$= 2$ rt. $\angle^s$, the sum of the $\angle^s$ of a $\triangle$.

Ex. 10. *Two circles touch one another and from the point of contact straight lines are drawn to the extremities of a diameter of one of them: prove that these two straight lines meet the other circle at the extremities of a parallel diameter.*



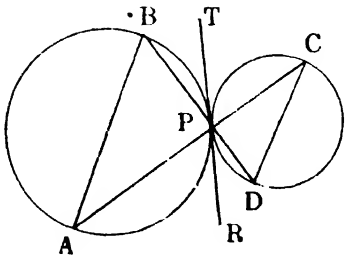


Fig. 1 Fig. 2

Let two circles touch at the pt. P internally in Fig. 1 and externally in Fig. 2 and let AB be a diameter of one of them; let AP, BP be joined to

cut (in Fig. 1), and be produced to cut (in Fig. 2), the other circle at C, D respectively.

Join CD.

It is reqd. to prove that CD is a diameter of the second circle and is parallel to AB.

Construction. At P draw the common tangent TPR to the two circles.

Proof. Now, the $\angle APB$ = a rt. $\angle$, being in semi-circle.

= the $\angle CPD$ [the same $\angle$ in Fig. 1 and vertically opp. $\angle$' in Fig. 2]

$\therefore$ CD is a diameter of the second circle.

Again, in Fig. 1, the $\angle TPA$ = the $\angle PBA$

[in the alt. segment of the circle APB]

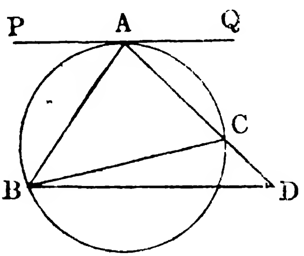
or, the $\angle CDP$ [in the alt. segment of the circle CPD]

$\therefore$ the $\angle CDP$ = the $\angle PBA$, but these are corresponding $\angle$'.

$\therefore$ AB and CD are parallel.

Similarly, it may be proved in Fig. 2 also that AB and CD are parallel.

Ex. 11. *AB and AC are the chords of a circle ABC and BD, a straight line through B drawn parallel to the tangent at A, intersects AC produced in D. Prove that AB is a tangent to the circum-circle of the triangle BCD.*



Let AB and AC be two chords of the circle ABC, the line PAQ be the tangent to the circle at the pt. A and the straight line BD drawn through B $\parallel$ to the tangent PAQ intersect AC produced in D. Join BC.

It is reqd. to prove that AB is a tangent to the circum-circle of the $\triangle BCD$.

Proof. Now the $\angle ADB$ = the alternate $\angle DAQ$ [$\because$ PQ and BD are $\parallel$]

= the $\angle ABC$ in the alternate segment

[$\because$ PQ is a tangent and AC the chord of contact]

$\therefore$ the $\angle ADB$ *i.e.* the $\angle CDB$ = the $\angle ABC$.

$\therefore$ the line BA drawn from the extremity of the chord CB of the circum-circle of the $\triangle BDC$ makes an angle with BC equal to the $\angle BDC$ in the alternate segment of the circum-circle.

$\therefore$ AB is a tangent to the circum-circle of the $\triangle BDC$.

[Converse of Theorem 11]

Exercise III

1. If two circles intersect, prove that the tangents to the two circles from any point on their common chord produced, are equal.

2. Prove that the common chord of two intersecting circles, when produced, bisects their common tangent.

3. If two circles intersect, and two chords are drawn one in each circle through any point in their common chord, or that chord produced, prove that the extremities of these chords are concyclic.

4. If two circles touch one another, prove that any straight line drawn through the point of contact cuts off similar segments from the two circles.

[Segments of circles containing equal angles are said to be *similar*]

5. Two circles intersect at A and B, through any point P on the circumference of one of them, chords PA and PB are drawn to cut the other circle at C and D; show that CD is parallel to the tangent at P.

6. If two circles touch one another, either internally or externally, and through A, the point of contact, two chords ABC and ADE (or BAC and DAE) are drawn to cut the two circles at B, D and C, E respectively, prove that BD and CE are parallel.

7. AB is a common chord of two circles, one of which passes through O, the centre of the other; prove the OA bisects the angle between the common chord and the tangent to the first circle at A.

8. Two circles touch internally and a chord of the greater is a tangent to the smaller; prove that the chord is divided at its point of contact into segments subtending equal angles at the point of contact of the circles.

9. Prove that the perpendicular drawn from the middle point of an arc of a circle, to the chord of the arc, is equal to the perpendicular drawn from the same point to the tangent at either extremity of the chord.

10. AOB is a diameter of a circle whose centre is O, and C is the middle point of OA. Chord PCQ is drawn perpendicular to OA. Prove that the triangle PBQ is equilateral, and $PQ^2 = 3OA^2$.

11. Two tangents TA and TB are drawn to a circle whose centre is O. Another circle is drawn through T and touching AB at A, and OA is produced to meet this circle again at D. Prove that OA is equal to AD.

12. From a point P, tangents PA, PC are drawn to a circle, and a secant PBD cuts the circle at B and D. Prove that in the quadrilateral $ABCD$, (i) $AB.CD = AD.BC$,

(ii) $AC.BD = 2AB.CD$.

13. A chord AB of a circle is produced on both sides to P and Q, such that $AP = BQ$, and from P and Q, tangents PX QY are drawn to the circle on opposite sides of AB, X and Y being the points of contact. Prove that the straight line XY bisects the chord AB.

14. Show how to construct a triangle having given the base, the vertical angle, and the ratio of the other two sides.

CHAPTER IV

PRACTICAL CONSTRUCTIONS

4·1. In the practical field construction of acute geometrical figures is of great importance. Of the different kinds of construction students have already become familiar with the easier ones like *the bisection of angles and straight lines, the drawing of a perpendicular to a given straight line at a given point in it or from a given point outside it, construction of an angle equal to a given angle at a given point in a given straight line, the drawing of a straight line through a given point parallel to a given straight line etc.*

In harder cases of construction, bisection of straight lines and angles, drawing of perpendiculars etc. are frequently resorted to. In these cases mere mention of what is to be done will do. How this is done need not be explained as the methods of these easy constructions are only too well known. Still the traces of these constructions should be left on the figure. An illustration will make the point clear. Suppose it is required to draw a tangent to a circle whose centre is O at a given point P on its circumference. Here it will be sufficient to say, "Draw PT perpendicular to OP". It is not necessary to state with what radius and where the arc of a circle is to be described. But the traces of the construction of lines and arcs of circles necessary to draw the perpendicular must be left on the figure.

In Practical Geometry, the figures drawn should be neat and accurate as far as practicable. To make the figure neat and clean, lines should be drawn very carefully with the fine point of a hard pencil, and greatest care should be taken to see that there should be no occasion for erasing any line badly drawn. The traces of construction should be clear and well-defined.

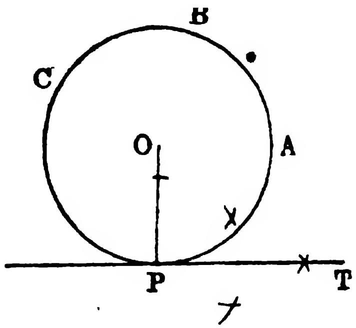
4·2. Construction of Tangents.

Problem 1

Draw a tangent to a circle at a given point on the circumference.

Let ABC be a circle whose centre is O and P the given point on the circumference.

It is required to draw a tangent to the circle ABC at the point P.



Construction. Join OP.

At P draw PT perp. to OP.

Then PT shall be tangent to the circle ABC at the point P.

Proof. O is the centre of the circle ABC and OP is its radius.

Now, the str. line PT is perp. to the radius OP.

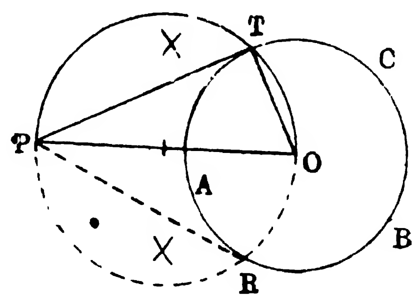
∴ PT is tangent to the circle ABC at the pt. P.

Problem 2

Draw a tangent to a circle from a given external point.

Let O be the centre of the circle and P a point outside it.

It is required to draw a tangent to the circle ABC. from the point P.



Construction. Join OP. On OP as diameter describe a semi-circle intersecting the circle ABC at the point T. Join PT.

Then shall the str. line PT be tangent to the circle ABC.

Proof. Join OT.

Now, the $\angle OTP$ being in a semi-circle is a rt. $\angle$ and OT is a radius.

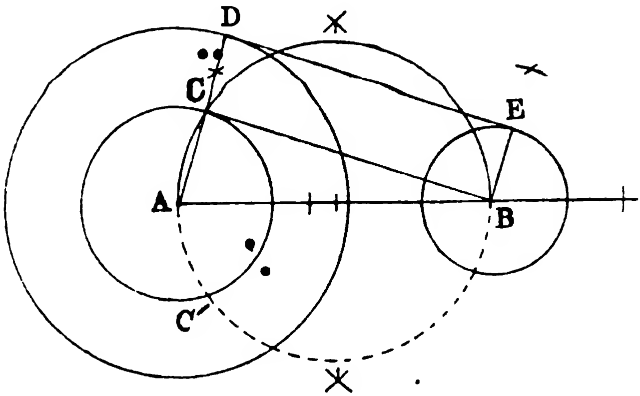
∴ the straight line PT being at rt. $\angle^s$ to the radius OT is tangent to the circle ABC.

N. B. If the complete circle be drawn on OP as diameter, it will intersect the given circle at another point, say R. If PR be joined a second tangent to the circle ABC will be obtained. Hence two tangents can be drawn to a circle from a given external point.

Problem 3

Draw a **direct** *common tangent to two given circles.*

Let A be the centre of the greater circle and a its radius and B be the centre of the smaller circle and b its radius.



It is reqd. to draw a common tangent to these two circles.

Construction. Join AB.

With centre A and radius $(a-b)$, the difference of the radii of the two circles, describe a circle. On AB as diameter another circle which cuts this circle at the point C. Join AC and produce it to meet the larger circle at D. Through B draw BE in the same direction as, and parallel to, AD to meet the smaller circle at E. Join DE.

Then shall DE be the direct common tangent to the two given circles.

Proof. Since $AD=a$ and $AC=a-b$, $\therefore$ $CD=b$.

$\therefore$ CD and BE are equal and parallel.

$\therefore$ the figure $BCDE$ is a parallelogram and the $\angle CDE=$ the corresponding $\angle ACB=$ a rt. $\angle$, being in a semi-circle.

$\therefore$ $BCDE$ is a rectangle.

$\therefore$ each of the $\angle^s$ BED, CDE is a rt. $\angle$ and since AD, BE are the radii of the two given circles, $\therefore$ DE is tangent to both the given circles.

Hence DE is a direct common tangent to the two given circles.

Note 1. The circle described on AB as diameter will intersect the circle with centre A and radius $(a-b)$ at also another pt. C' besides the pt. C. Hence a second direct common tangent can be drawn on the other side of AB.

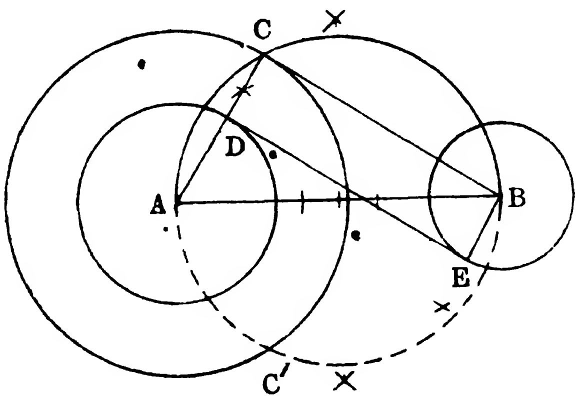
Note 2. With the same construction as given above direct common tangents to two given circles can be drawn even when they intersect one another.

Note 3. To draw direct common tangents to two equal circles, proceed as follows :—

Join the centres of the two given circles, and through their centres draw two radii on the same side of this joining line perpendicular to it. If the two extremities of the two radii so drawn on each side be joined, two direct common tangents to the two circles will be obtained.

Problem 4

Draw a **transverse** *common tangent to two given circles.*



Let A, B be the centres of the two given circles and a, b be their radii respectively.

It is reqd. to draw a transverse common tangent to the two given circles.

Construction. Join AB.

With centre A and radius equal to $(a+b)$, the sum of the two given circles, draw a third circle. From B draw a tangent BC to the third circle (*i.e.*, on AB as diameter describe a circle cutting the third circle at C. Join BC). Join AC cutting the first circle at the pt. D. Now through B draw BE parallel and on the side opposite to AC. Join DE.

Then DE shall be the transverse common tangent to the two given circles.

Proof. Since $AC=a+b$ and $AD=a$, $\therefore$ $CD=b=BE$ and BE, CD are parallel (by construction).

$\therefore$ BE, CD are both equal and parallel.

Hence $BEDC$ is a parallelogram.

Again, $\angle C =$ a rt. $\angle$, being in a semi-circle.

$\therefore$ $BEDC$ is a rectangle.

$\therefore$ the $\angle ADE =$ the $\angle BED =$ a rt.-$\angle$.

Since AD, BE are radii, DE is a common tangent to both the circles.

N. B. Since the circle described on AB as diameter, intersects the circle with centre A and radius $(a+b)$ at a second pt. C', a second transverse common tangent to the two given circles can also be drawn.

If the two circles intersect one another, they have no transverse common tangent.

4·3. Construction of regular figures in or about a given circle.

A plane rectilinear figure whose sides are equal to one another and whose angles also are equal to one another is called a regular figure. From the definition of a regular figure an equilateral triangle and a square are regular figures. A regular pentagon is a five-sided figure all of whose sides are equal to one another and angles are also equal to one another.

The number of sides of a rectilinear figure is equal to the number of its angles *i.e.*, if the number of sides be equal to 8, the number of its angles will also be 8. Again, if the number of sides of a rectilinear figure be known, the sum of its angles is also

known. Hence the magnitude of an angle of a regular rectilinear figure with a given number of sides may be easily determined. If the sum of the angles of a regular polygon with a given number of sides be divided by the number of sides of the polygon, the magnitude of an angle of the regular polygon can be found *e.g.* each angle of a regular pentagon is 108°, that of a regular hexagon is 120°, and so on.

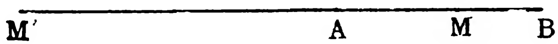
Again, the number of angles subtended at the centre of a polygon by its sides is equal to the number of its sides. Whatever may be the number of sides of a polygon, the sum of the angles subtended by its sides at the centre is always equal to 360°. Hence, if 360° be divided by the number of sides of a regular polygon, it will give the magnitude of an angle subtended by a side of the regular polygon at its centre. If the magnitude of this angle be subtracted from 180°, it will give the magnitude of an angle of the regular polygon.

The centre of a regular polygon inscribed or circumscribed, in or about a circle is identical with the centre of the circle. Hence to inscribe or circumscribe a regular polygon in or about a circle, at first the angle subtended by a side of the polygon at its centre is determined by dividing 360° by the number of sides of the polygon and then an angle equal to this angle is drawn at the centre of the circle. The length of the straight line joining the two points at which the arms of the angle so drawn cut the circle gives the length of a side of the polygon. If chords equal in length to this side be successively placed inside the circle, the polygon will be inscribed in the circle as required. If tangents to the circle be drawn at the extremities of these chords, the circumscribed polygon will be obtained.

It is easy to inscribe or circumscribe an equilateral triangle, a square or a regular hexagon in or about a circle as the angles subtended by a side of these regular figures at the centre are 120°, 90° and 60° respectively and these are easy to draw. But the angle subtended at the centre by the side of a regular pentagon is $\frac{360^\circ}{5}$ or 72° and to draw this angle it is necessary to know what medial section is. Hence before proceeding to inscribe or circumscribe regular figures in or about a circle, we shall define 'medial section' and discuss about it.

Medial Section

Definition. A straight line is said to be divided in **medial section** when the rectangle contained by the whole line and one part is equal to the square on the other part ; and the point of division is said to be the *point of medial section.* A straight line may be divided (i) internally or (ii) externally in medial section.

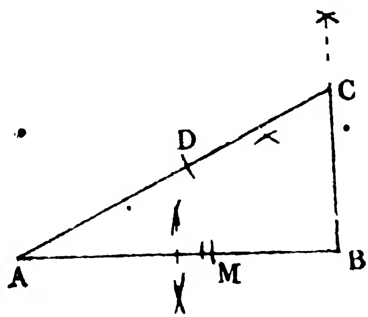


In the above figure (i), if $AB.BM = AM^2$, then the line AB is said to be divided internally in medial section at the pt. M ; and (ii) if $AB.BM' = AM'^2$, then the line AB is said to be divided externally in medial section at the pt. M'.

Problem 5

Divide a given straight line **internally** *in medial section.*

Let AB be the str. line to be divided in medial section.



Construction. At the point B in AB draw BC perp. to AB. Take $BC = \frac{1}{2}AB$. Join AC. From CA cut off $CD = CB$. Now, with centre A and radius equal to AD draw an arc cutting AB at the pt. M.

Then shall AB be divided internally at M in the medial section.

Proof. Let $AB = a$ and $AM = x$. $\therefore$ $BM = a - x$,

$$BC = \tfrac{1}{2}a,\ AC = AD + DC = AM + BC = x + \tfrac{1}{2}a.$$

Now, since ABC is a rt. $\angle^d$ $\triangle$,

$$AC^2 = AB^2 + BC^2 \quad i.e.,\ (x + \tfrac{1}{2}a)^2 = a^2 + (\tfrac{1}{2}a)^2.$$

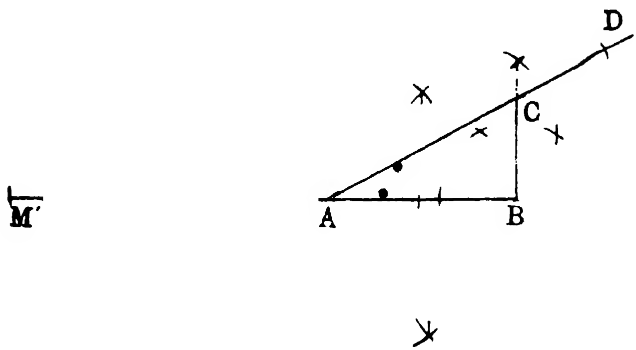
$\therefore\ x^2 + ax = a^2, \quad \therefore\ x^2 = a^2 - ax = a(a - x)$

$$i.e.,\quad AM^2 = AB.BM.$$

$\therefore$ the str. line AB is divided internally at M in medial section.

Problem 6

Divide a given straight line **externally** *in medial section.*



Let AB be the str. line to be divided externally in medial section.

Construction. At B in the str. line AB draw a perp. to AB and take $BC = \frac{1}{2}AB$. Join AC and produce it to D making $CD = CB$.

Now with centre A and radius equal to AD draw an arc of a circle which intersects BA produced at M'.

Then shall the str. line AB be divided externally in medial section at M'.

Proof. Let $AB = a$ and $AM' = x$.

$\therefore\ BM' = x + a,\ AD = x,\ CD = CB = \frac{1}{2}a.$

$$AC = AD - CD = x - \tfrac{1}{2}a.$$

Now, since ABC is a rt $\angle^d\ \triangle$,

$$AC^2 = AB^2 + BC^2 \quad i.e.,\ (x - \tfrac{1}{2}a)^2 = a^2 + (\tfrac{1}{2}a)^2.$$

$\therefore\ x^2 - ax = a^2. \quad \therefore\ x^2 = ax + a^2 = a(x + a)$

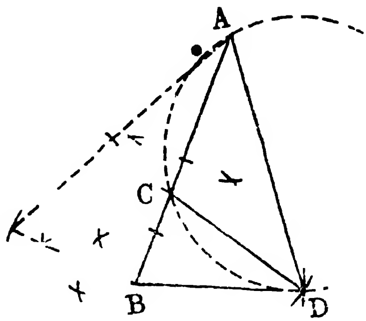
$$i.e.,\quad AM'^2 = AB.BM'.$$

$\therefore$ the str. line AB is divided externally at M' in medial section

Problem 7

Construct an isosceles triangle having each of the angles at the base double of the vertical angle.

It is reqd. to describe an isosceles triangle each of whose base angles is double of the vertical angle.



Construction. Take any str. line AB. Divide AB at C so that $AB \cdot BC = AC^2$, *i.e.*, divide AB internally at C in media section.

With centres B and C and radius equal to AC draw two arcs of a circle and let them cut each other at D.

Join BD, AD.

Then ABD shall be the reqd. $\triangle$.

Proof. Join CD and circumscribe a circle about the $\triangle ACD$.

Now $AB \cdot BC = AC^2 = BD^2$.

$\therefore$ BD touches the circle ACD at the pt. D.

$\therefore$ the $\angle BDC$ = the $\angle CAD$ in the alternate segment.

Again, by construction $AC = CD$, $\therefore$ the $\angle CAD$ = the $\angle CDA$,

$\therefore$ the $\angle BDA$ = the $\angle BDC$ + the $\angle CDA$ = the $\angle CAD$ + the $\angle CDA = 2\angle CAD$ *i.e.* $2\angle BAD$.

Again, since $BD = CD$,

$\therefore$ the $\angle DBC$ = the $\angle DCB$ = the $\angle CAD$ + the $\angle CDA$

$= 2\angle BAD$, $\therefore$ $AB = AD$.

[$\because$ $\angle DBA$ *i.e.* $\angle DBC = \angle BDA$, each being $= 2\angle BAD$]

$\therefore$ the $\triangle ABD$ is isosceles and each of its base $\angle^s$ ABD, ADB is double of the vertical $\angle A$.

N. B. Because $\angle B = \angle D = 2\angle A$,

$\therefore$ $\angle A + \angle B + \angle D = \angle A + 2\angle A + 2\angle A = 5\angle A$.

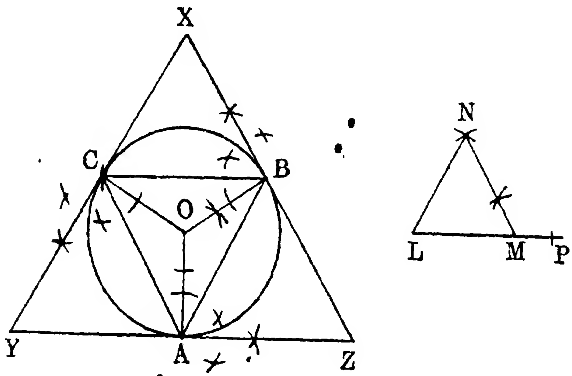
$\therefore$ $5\angle A=180°$ [the sum of the $\angle$' of the $\triangle ABD$]

$\therefore$ $A=36°$ and $\angle B=\angle D=72°$.

$\therefore$ now to draw an $\angle=72°$, if a triangle be described in the above manner the $\angle ABD$ will be $72°$.

Problem 8

Construct a regular figure of three sides (that is, an equilateral angle) (i) in, or (ii) about a given circle.



Let O be the centre of the given circle.

It is reqd. to (i) inscribe an equilateral $\triangle$ in the circle and (ii) circumscribe an equilateral $\triangle$ about the circle.

(i) Construction. Construct an equilateral $\triangle LMN$. Produce LM to P. Take any radius OA of the circle and at O, the centre of the circle, draw the $\angle AOB=$ the $\angle NMP$, so that OB cuts the circle at B. Join AB. Place a chord BC equal to AB inside the circle. Join AC.

Then $\triangle ABC$ shall be the reqd. equilateral $\triangle$ inscribed in the circle.

Proof. The $\triangle LMN$ is equilateral, $\therefore$ each of its $\angle^s=60°$.

$\therefore$ the $\angle NMP=120°$.

$\therefore$ the $\angle AOB$ drawn at $O=120°$.

Again, the chord $BC=$ the chord AB.

$\therefore$ the $\angle BOC$ = the $\angle AOB = 120°$.

$\therefore$ the remaining $\angle AOC = 120°$

[$\because$ sum of the $\angle^s$ at $O = 360°$]

Now, since the three $\angle^s$ AOB, AOC and BOC at the centre are equal to one another, the three chords, AB, AC, BC subtending these three angles are equal to one another.

$\therefore$ $AB = AC = BC$.

$\therefore$ the $\triangle$ inscribed in the given circle is equilateral.

(ii) Construction. As before draw the $\angle$ and placing the chord BC equal to AB in the three pts. A, B, C on it. Draw three tangent A, B, C *i.e.*, perp. to OA, OB, OC. Let these t one another at the pts. X, Y, Z.

Then $\triangle XYZ$ shall be the reqd. equilateral about the circle.

Proof. Because ZY and ZX are tanger each of the $\angle^s$ OAZ, OBZ = a rt. $\angle$

[$\because$ tangent and rad

$\therefore$ $AOBZ$ is a cyclic quadrilateral

$\therefore$ the $\angle AOB$ and the $\angle AZB$ are supplem

But the $\angle AOB = 120°$ (by construction)

$\therefore$ the $\angle AZB$ *i.e.* the $\angle YZX = 60°$.

Similarly, it may be proved each of the $\angle^s$ XYZ, $ZXY = 60°$.

$\therefore$ the $\triangle XYZ$ is equilateral as the sides are tangents to the circle it is circumscribed about it.

Problem 9

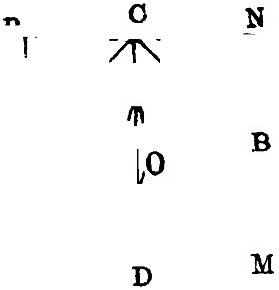
Construct a regular figure of four sides (that is a square) (i) in or (ii) about a given circle.

Let O be the centre of the given circle.

It is reqd. to (i) inscribe a square in the circle,

(ii) circumscribe a square about the circle.

Construction. Take a diameter AOB of the circle and at the centre O take another diameter COD at rt. $\angle^s$ to AOB *i.e.* bisect AB at rt. $\angle^s$.



(i) Join AC, CB, BD and DA.

Then $ACBD$ will be the reqd. regular quadrilateral *i.e.* square inscribed in the circle.

Proof. Because each of the $\angle^s$ AOC, COB, BOD and DOA at the centre of the circle being a rt. $\angle$, the chords AC, CB, BD and DA are equal to one another.

Again each of the $\angle^s$ ACB, CBD, BDA and $\angle DAC$ being in a semi-circle, is a rt. $\angle$.

$\therefore$ $ACBD$ is a reqular quadrilateral, or square, inscribed in the circle.

(ii) At the pts. A and B draw two str. lines $\parallel$ to CD and at the pts. C and D draw another two str. lines $\parallel$ to AB and let them intersect at the pts. L, M, N, P.

Then $LMNP$ shall be the reqd. regular quadrilateral or square circumscribed about the circle.

Proof. LM is $\parallel$ to AB and AL and BM being each $\parallel$ to CD are $\parallel$ to one another.

$\therefore$ $ALMB$ is a parallelogram, $\therefore$ $LM = AB$.

Similarly, $PN = AB$. $\therefore$ $LM = PN = AB$.

Similarly, $PL = NM = CD$. But $AB = CD$

(diameters of the circle

$\therefore$ $LM = MN = NP = PL$.

Again, by construction $COBN$ is a parallelogram and the $\angle COB =$ a rt. $\angle$. $\therefore$ $\angle N$ is a rt. $\angle$. Similarly the $\angle^s$ L, M, P are also rt. $\angle^s$.

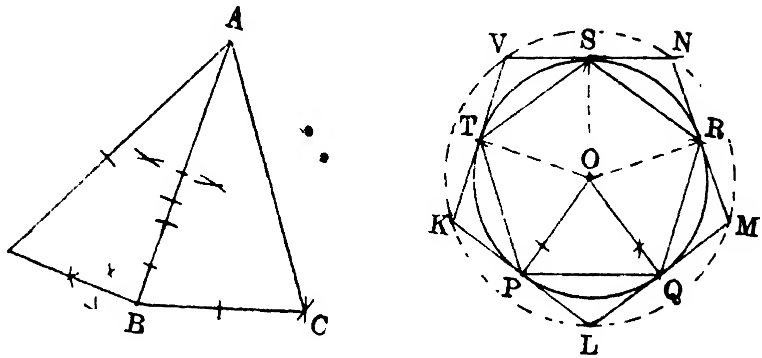
$\therefore$ the sides and angles of $LMNP$ being equal to one another, the Fig. $LMNP$ is a square.

Now the $\angle COB$ of the parallelogram $COBN$ being a rt. $\angle$ each of the $\angle^s$ OCN and OBN is a rt. $\angle$.

$\therefore$ PN and MN touch the given circle at the pts. C, B.
Similarly, LM and LP are tangents to the circle.
$\therefore$ $LMNP$ is the reqd. square circumscribed about the circle.

Problem 10

Construct a regular pentagon (i) in or (ii) about a given circle.



Let O be the centre of the given circle.

It is reqd. to (i) inscribe a regular pentagon in the circle,
and (ii) circumscribe a regular pentagon about the circle

(i) Construction. Describe an isosceles triangle ABC having each of the $\angle^s$ ABC, ACB double of the vertical $\angle BAC$.

Take any radius OP of the circle and at the centre O of the circle draw the $\angle POQ$ equal to the $\angle ABC$, so that the arm OQ cuts the circle at Q. Join PQ. Place three chords QR, RS, ST successively in the circle each equal to PQ. Join TP.

Then $PQRST$ shall be the reqd. regular pentagon inscribed in the circle.

Proof. By construction, the $\angle ABC = 72°$ (Prob. 7)

$\therefore$ the $\angle POQ = 72°$

Since PQ, QR, RS, ST are equal chords of a circle,

$\therefore$ each of the $\angle^s$ POQ, QOR, ROS, SOT at the centre of the circle $= 72°$.

$\therefore$ the sum of these 4 angles $= 4 \times 72^\circ = 288^\circ$.

But the sum of the $\angle^s$ at the pt. $O = 360^\circ$.

$\therefore$ the remaining $\angle TOP = 360^\circ - 288^\circ = 72^\circ$.

$\therefore$ the chord TP is equal to the other chords.

$\therefore$ the sides of the pentagon $PQRST$ are equal to one another.

Again, the 5 Δ^s POQ, QOR etc. are equal in all respects and each of them is isosceles.

$\therefore$ the $\angle OPQ =$ the $\angle OQP$. Hence, each of the vertical $\angle^s$ of the Δ^s being 72°, their base angles are all equal to one another.

$\therefore$ the $\angle PQR =$ the $\angle PQO +$ the $\angle OQR = 2\angle PQO$.

$\therefore$ the $\angle^s$ P, Q, R etc. of the pentagon being double the base-angle of these Δ^s are equal to one another.

$\therefore$ the sides and angles of the pentagon $PQRST$ being equal to one another, the pentagon is a regular one.

(ii) At the points P, Q, R, S, T of the circle, draw five perpendiculars KL, LM, MN, NV, VK to the radii OP, OQ, OR, OS, OT respectively.* The pentagon $KLMNV$ they have formed by meeting at the pts. K, L, M, N, V shall be the required regular pentagon circumscribed about the circle.

Proof. Since by construction KL, LM, MN, NV, VK are perps. to the radii at the pts. P, Q, R, S, T respectively on the circle, they are tangents to the circle.

Now, by construction, the $\angle OPL +$ the $\angle OQL = 2$ rt. $\angle^s$.

$\therefore$ the four points P, O, Q, L are concyclic.

$\therefore$ the $\angle L$ is the supplement of the $\angle POQ$.

Similarly, the $\angle^s$ M, N etc. are the supplements of their respective opposite $\angle^s$ QOR, ROS etc. at the pt. O.

But the five angles $\angle^s$ POQ, ROS etc. at the pt. O are equal to one another.

*The traces of construction of the perpendiculars have not been left on

$\therefore$ the five angles $\angle L$, $\angle M$, $\angle N$ etc. being supplementary of these equal angles are equal to one another.

Again, each of the Δ^s LPQ, QMR etc. is an isosceles Δ.

[$\because$ the two tangents drawn from each of the points L, M, N etc. to the circle are equal.]

$\therefore$ the $\angle LPQ$ = the $\angle LQP$, the $\angle MQR$ = the $\angle MRQ$,......

But the $\angle^s$ L, M, N etc. of the Δ^s LPQ, MQR, NRS etc. are equal to one another. (Proved)

$\therefore$ the $\angle^s$ LPQ, LQP, MQR, MRQ etc are equal to one another.

Now, in the two Δ^s LPQ and MQR, the $\angle L$ = the $\angle M$, the $\angle LPQ$ = the $\angle MQR$ and $PQ = QR$. $\therefore$ the two Δ^s are equal in all respects.

$\therefore$ $PL = MR$ and $LQ = QM$.

$\therefore$ $LM = 2QM$. Similarly, $MN = 2MR$.

But $QM = LQ = LP = MR$. $\therefore$ $LM = MN$.

$\therefore$ the sides LM, MN etc. of the pentagon are all equal to one another.

$\therefore$ $KLMNV$ is a regular pentagon and it has been circumscribed about the given circle.

Alternative Method

Without drawing perpendiculars to the radii through the five pts. P, Q, R, S, T, the construction for circumscribing a regular pentagon about a circle may be shortened in the following manner.

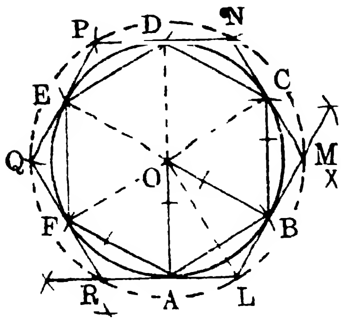
As before draw the $\angle POQ = 72°$ at the centre O of the circle. Draw perps. to the radii OP, OQ through the pts. P, Q. Let them intersect at L. With centre O and radius OL draw another circle. Let LP, LQ produced meet this circle at K, M respectively. Now, place the chords MN, NV equal to LM successively in this circle and join VK. Then shall $LMNKV$ be the required regular pentagon about the given circle.

The proof is left to the students as an exercise.

Problem 11

Construct a regular hexagon (i) or (ii) about a given circle.

Let O be the centre of the given circle.



It is reqd. to (i) inscribe or (ii) circumscribe a regular hexagon (i) in or (ii) about this circle.

(i) Construction. Take any radius OA of the circle and place successively five chords AB, BC, CD, DE, EF in the circle, each equal to OA.

Join AF.

Then shall $ABCDEF$ be the regular hexagon inscribed in the given circle.

Proof. Join OA, OB, OC, OD, OE, OF.

Now the five $\triangle^s$ AOB, BOC etc. each being an equilateral one, have each of their angles $= 60°$.

Again, the sum of the $\angle^s$ at $O = 360°$.

But the sum of five $\angle^s$ AOB, BOC etc. $= 5 \times 60° = 300°$.

$\therefore$ the remaining $\angle AOF = 360° - 300° = 60° =$ the $\angle AOB$.

$\therefore \quad AF = AB = OA$.

i.e. the $\triangle AOF$ is also equilateral.

$\therefore$ the six sides of the hexagon are equal to one another and each of the six $\angle^s$ A, B etc. being $60° + 60°$ *i.e.* $120°$, they are equal to one another.

Hence the hexagon $ABCDEF$ is regular and it is inscribed in the given circle.

(ii) Construction. Take any radius OA of the given circle and place a chord AB in the circle equal to OA. Join OB.

At the points A and B draw perps. to OA and OB and let them intersect at L. Join OL.

Now, with centre O and radius OL draw another circle. Let LA, LB produced cut the circle at the pts. R and M. Beginning

with the pt. M, successively place three chords MN, NP, PQ each equal to LM in their circle. Join QR. Then shall $LMNPQR$ be the reqd. regular hexagon circumscribed about the given circle.

Proof. By construction, the ΔOAB is equilateral.

[$\because$ $OA = OB = AB$]

$\therefore$ the $\angle AOB = 60°$.

Again, the rt.-$\angle^d$ Δ^s OAL, OBL are equal in all respects

[$\because$ $\angle^s$ A and B are rt. $\angle^s$, $OA = OB$ and the hypotenuse OL is common]

$\therefore$ the $\angle LOB =$ the $\angle LOA = 30°$.

Now, LM is a chord of the outer circle and OB is perp. to it.

$\therefore$ B is the middle point of LM.

the Δ^s OBL, OBM are equal in all respects.

[$\because$ $BL = BM$, OB is common and the rt. $\angle OBL =$ rt. $\angle OBM$]

$\therefore$ the $\angle MOB =$ the $\angle LOB = 30°$, $\therefore$ the $\angle LOM = 60°$.

Similarly, the $\angle LOR = 60°$, and since the chords LM, MN, NP, PQ of the outer circle are equal to one another, the $\angle^s$ subtended by them at the centre are equal to one another.

i.e., the $\angle MON =$ the $\angle NOP = \angle POQ =$ the $\angle LOM = 60°$ $=$ the $\angle LOR$.

But the whole $\angle$ round the pt. $O = 360°$.

$\therefore$ the $\angle ROQ = 360° - 5 \times 60° = 60°$

$\therefore$ the chord $RQ =$ the chord LM *i.e.* all the sides of the hexagon $LMNPQR$ are equal.

Again, these sides being equal chords of the outer circle are equidistant from the centre O of the circle and this distance $= OA =$ the radius of the given circle.

$\therefore$ the feet of the perps. [A, B and C, D, E, F say] from the centre O upon the sides of the hexagon lie upon the given circle.

$\therefore$ the sides of the hexagon $LMNPQR$ touch the given circle at the feet of the perpendiculars A, B, C, D, E, F *i.e.* the hexagon $LMNPQR$ is circumscribed about the given circle.

Now $OALB$ is a cyclic quadrilateral.

[$\because$ $\angle A$ and $\angle B$ are rt. $\angle^s$]

$\therefore$ the $\angle ALB$ + the $\angle AOB = 180°$, but the $\angle AOB = 60°$.

$\therefore$ the $\angle ALB$ *i.e.* the $\angle RLM = 120°$.

Similarly, each $\angle$ of the hexagon $LMNPQR = 120°$.

$\therefore$ the sides and angles of the hexagon being equal to one another, it is regular and is circumscribed about the given circle.

Alternative method.

From any pt. A on the given circle, successively place five chords AB, BC, CD, DE, EF each equal to the radius of the circle.

Now, at the six pts. A, B, C, D, E, F draw six tangents (*i.e.* draw six perps. to the radii through these pts.) to the circle and let them intersect at the six pts. L, M, N, P, Q, R.

Then shall $LMNPQR$ be the regular hexagon circumscribed about the circle.

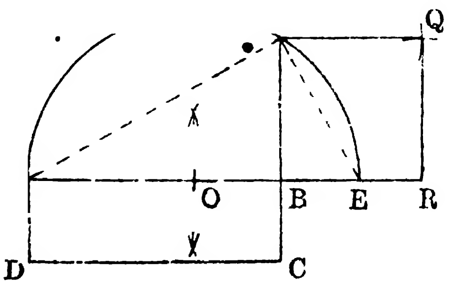
The proof is left to the students as an exercise.

N. B. In the First Method of drawing the circumscribed hexagon, the construction is comparatively short, although the proof is long. In Second Method, the proof is easier, but the construction is rather long.

4·4. Construction of a square equal in area to a given polygon.

Problem 12

Construct a square equal in area to the given rectangle.



Let $ABCD$ be the given rectangle.

It is reqd. to describe a square equal in area to the rectangle $ABCD$.

Construction. Produce the side AB of the rect. $ABCD$ to E so that $BE = BC$.

Draw a semi-circle on AE as diameter. Produce CB to meet the semi-circle at P. Cut off BR from BE produced making it equal to BP. Now with centres P, R and radius equal to BP draw two arcs intersecting at the pt. Q. Join PQ, RQ.

Then shall $PBRQ$ be the reqd. square.

Proof. It can be easily proved that $PBRQ$ is a square.

Join AP, PE.

Now, the $\angle APE$ = a rt. $\angle$ (being in a semi-circle and PB is drawn from the rt. $\angle$ perp. to the hypotenuse AE),

$\therefore \quad PB^2 = AB \,.\, BE = AB \,.\, BC,$

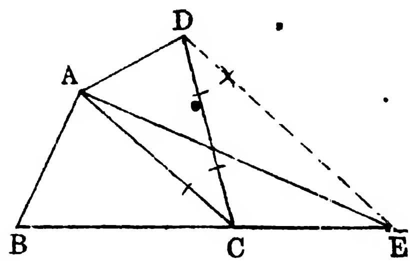
i.e. the square $PBRQ$ = the rect. $ABCD$ in area.

N. B. Construction of a square equal in area to a given triangle.

From the vertex of the given triangle draw a perp. to the base. The area of the rectangle with sides equal to this perpendicular and half the base is equal to the area of the given triangle. Hence, by the above Problem, a square drawn equal in area to this rectangle will be the square required.

Problem 13

Construct a triangle equal in area to a given quadrilateral.



Let $ABCD$ be the given quadrilateral.

It is reqd. to draw a triangle equal in area to the quadrilateral $ABCD$.

Construction. Join AC. Through D draw $DE \parallel$ to AC to meet BC produced at E. Join AE.

Then the $\triangle ABE$ shall be the reqd. $\triangle$.

Proof. The $\triangle^s$ ACD, ACE stand on the same base AC and between the same parallels AC, DE. $\therefore$ the $\triangle ACD =$ the $\triangle ACE$.

To each of them add the $\triangle ABC$.

$\therefore$ the quadrilateral $ABCD =$ the $\triangle ABE$.

N. B. By the method shown above, we can first reduce a pentagon to a quadrilateral of equal area and then we can draw a triangle equal in area to this quadrilateral. Hence when a polygon with any number of sides is given by gradually diminishing the number of its sides by one at a time in the manner shown above, we can ultimately draw a triangle equal to the given polygon in area ; and then by drawing a square equal to this triangle, we can draw a square equal in area to any given polygon. The construction of a square equal in area to a given pentagon is shown below by way of an example.

Problem 14

Construct a square equal in area to a given pentagon.

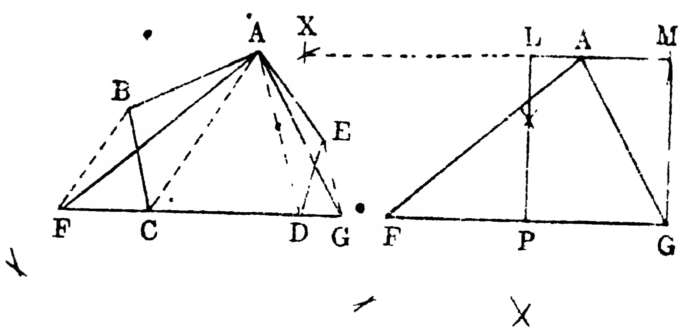


Fig. 1. Fig. 2

The required construction is here effected in three stages :—

(i) Construction of a triangle equal to the pentagon in area,

(ii) Construction of rectangle equal to the triangle so drawn,

and (iii) Construction of the square equal to the rectangle drawn.

If the work of construction be made with the same figure, it will render the figure untidy. So the construction at different stages is shown in three separate figures. In the first figure, a triangle equal to the pentagon, in the second, a rectangle equal to the triangle and in the third the square equal to the rectangle have been drawn.

Let $ABCDE$ be the given pentagon.

It is reqd. to describe a square equal to the pentagon $ABCDE$ in area.

Construction. (i) Join AC, AD. Throgh B draw $BF \parallel$ to AC and through E draw $EG \parallel$ to AD to meet DC and CD produced at F and G respectively. Join AF, AG. The $\triangle AFG$ is equal to the pentagon in area (Fig. 1).

(ii) Through A draw $AX \parallel$ to FG. Bisect FG at P. Through P draw PL perpendicular-bisector of FG to meet AX at L. From XL produced cut off $LM = PG$. Then $LPGM$ is a rectangle and its area is equal to that of the $\triangle AFG$ i.e. to that of the given pentagon (Fig. 2).

(iii) Take a str. line equal to LM and produce it to K. From MK cut off a length equal to MG of Fig. 2. On LG as diameter describe a semi-circle. At the pt. M draw MN perp. to LG cutting the semi-circle at N. (Fig. 3)

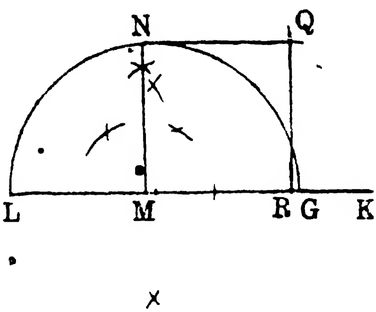


Fig.

The square described on MN is the reqd. square,

The pentagon $ABCDE$ = the $\triangle AFG$ = the rect. $LPGM$
= the square $MNQR$.

4·5. Construction of a mean proportional to two given straight lines.

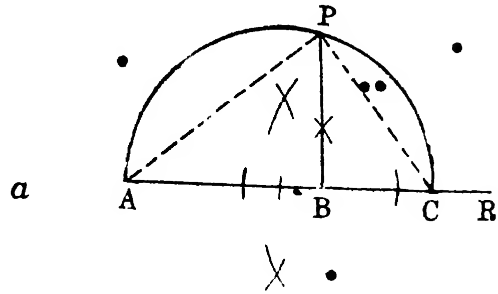
Problem 15

Find a mean proportional to two given straight lines.

Let a and b be the two given str. lines.

It is reqd. to find the mean proportional between the str. lines a and b.

Construction. Take any str. line AR and from AR cut off AB and BC equal to a and b respectively. On AC as diameter describe a semi-circle. At the pt. B draw BP perp. to AC to meet the semi-circle at P.



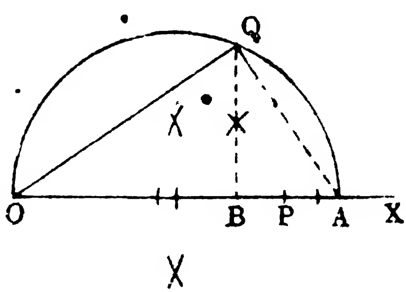
Then shall the str. line BP be the mean proportional between the str. lines AB and BC *i.e.* between the str. lines a and b.

Proof. Join AP, PC. The $\angle APC$ in the semi-circle = a rt. $\angle$ and from the rt. $\angle P$, PB is drawn perp. to the hypotenuse AC.

$$\therefore \quad PB^2 = AB \cdot BC = a \cdot b$$

i.e. PB is the mean proportional between a and b.

Alternative Method.



Construction. From a str. line OX cut off OA and OB equal to a and b. Suppose $OA > OB$. On OA as diameter describe a semi-circle. At the pt. B draw BQ perp. to OB to meet the semi-circle at Q. Now from OA cut off OP equal to OQ.

Then shall OP or OQ be the mean proportional between OA and OB.

Proof. Join OQ, QA.

Since the $\angle OQA$ in the semi-circle = a rt. $\angle$
= the $\angle QBO$
(by construction),

and the $\angle QOB$ is common,

$\therefore$ the Δ^s AOQ and QOB are equiangular and hence similar.

$\therefore \frac{OA}{OQ} = \frac{OQ}{OB}$. $\quad OA \cdot OB = OQ^2 = OP^2$.

$\therefore$ OP or OQ is the mean proportional between OA and OB.

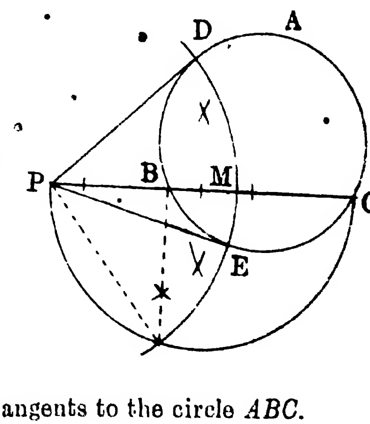
N. B. If it is required to find the mean proportional between two straight lines like OA and OB drawn from a pt. in the same direction, it is convenient to apply the Alternative Method and in the subsequent constructions this method has been resorted to.

Corollary. With the help of this problem we can draw a tangent to a given circle from an external point.

Let ABC be the given circle and P, a point outside it.

It is reqd. to draw a tangent to the circle ABC from P.

Construction. From the point P draw any secant PBC of the circle. Find the mean proportional PM between PB and PC. With centre P and radius equal to PM draw an arc of a circle to cut the given circle at the pts. D, E. Join PD, PE.



Then shall PD, PE be two tangents to the circle ABC.

Proof. Since PM is the mean proportional between PB and PC, $\therefore$ $PB \cdot PC = PM^2 = PD^2$ or PE^2.

Hence PD and PE are tangents to the circle ABC,

4·6. Illustrative Examples.

Ex. 1. *Divide a given straight line in a given ratio (i) internally or (ii) externally.*

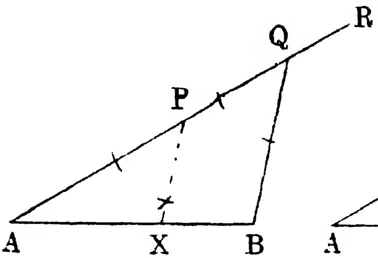


Fig. 1

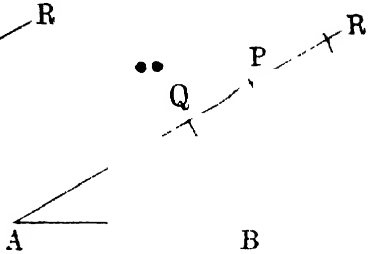


Fig. 2

Let AB be the given st. line and $m : n$ the given ratio.

It is reqd. to divide AB (i) internally or (ii) externally in the ratio $m : n$.

(i) Construction. Draw any str. line AR through A and from it cut off two successive lengths AP, PQ so that $AP = m$ and $PQ = n$ (Fig. 1). Join QB.

Now through P draw $PX \parallel$ to QB cutting AB at X.

Then shall AB be divided internally at X in the ratio $m : n$.

i.e. $AX : XB = m : n$.

Proof. Since PX and QB are parallel,

$\therefore$ $AX : XB = AP : PQ = m : n$.

(ii) Construction. Draw any str. line through AR and from it cut off AP equal to m. From PA cut off PQ equal to n. (Fig. 2). Join QB. Now through P draw $PY \parallel$ to QB to cut AB or BA produced at Y.

Then shall AB be divided externally at Y in the ratio $m : n$.

i.e. $AY : YB = m : n$.

Proof. Since PY and QB are parallal,

$\therefore$ $AY : YB = AP : PQ = m : n$.

Ex. 2. *Divide a given str. line AB internally at P, such that (i) $AP^2 = 2PB^2$, (ii) $AP^2 = 3PB^2$.*

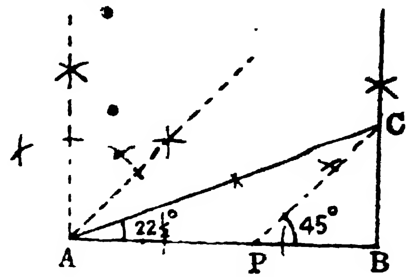
(i) Let AB be the given str. line.

It is reqd. to divide AB internally at P such that $AP^2 = 2PB^2$.

Construction. At A in AB draw the $\angle BAC = 22\frac{1}{2}°$ (At A draw a pe to AB and bisect the rt. $\angle$; then again bisect half the rt. $\angle$). At B dr perp. to AB and let it cut AC at C.
Now from BA cut off $BP = BC$.

Then shall AB be divided as reqd. at P,

i.e. AP^2 will be equal to $2BP^2$.



Proof. Join PC.

Now, by construction,

$BP = BC$, ∴ the $\angle BPC$ = the $\angle BCP = 45°$ [∵ the $\angle PBC = 90°$]

But the $\angle BPC$ = the $\angle CAP$ + the $\angle PCA$

i.e. $45° = 22\frac{1}{2}°$ + the $\angle PCA$.

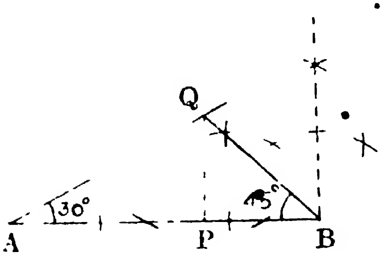
∴ the $\angle PCA = 22\frac{1}{2}°$ = the $\angle PAC$, ∴ $AP = PC$.

∴ $AP^2 = PC^2 = PB^2 + BC^2$ [∵ the $\angle PBC = 90°$]

$= 2PB^2$. [∵ $PB = BC$]

(ii) Let AB be the given straight line.

It is reqd. to divide AB internally at P such that $AP^2 = 3PB^2$.



Construction. At A in AB dra the $\angle BAQ = 30°$ (first draw an $\angle$ 60° of an equilateral $\triangle$ at A a then bisect it). At B make the $\angle AB$ $= 45°$ (first draw a perp. to AB at and then bisect the rt. $\angle$). Let A and BQ meet at Q. From Q dra the perp. QP on AB.

Then shall AB be divided reqd. at P,

i.e. AP^2 will be equal to $3PB^2$

Proof. The $\angle QPB = 90°$ and the $\angle QBP = 45°$, ∴ the $\angle PQB = 45°$.

∴ $QP = PB$.

Again, the $\angle QPA$ is a rt. $\angle$ and the $\angle QAP = 30°$.

∴ the hypotenuse = twice the smallest side QP.

i.e. $AQ = 2QP$.

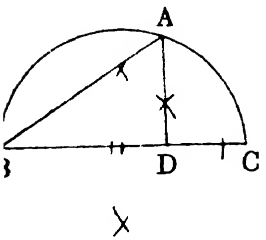
Now, since the $\angle APQ$ is a rt. $\angle$,

$\therefore \quad AP^2 = AQ^2 - QP^2 = (2QP)^2 - QP^2 = 3QP^2$

$= 3PB^2$. [$\because$ $QP = PB$, proved]

Ex. 3. *Find geometrically the value of* $\sqrt{6}$.

Take a str. line BC, 3 units in length and from it cut off $BD = 2$ units long.



Find the mean proportional BA between BC and BD.

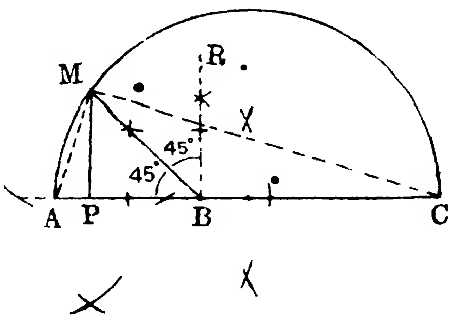
The str. line AB is the mean proportional between BC and BD.

$\therefore \quad AB^2 = BC \,.\, BD = 3 \,.\, 2$ sq. units $= 6$ sq. units

$\therefore \quad AB = \sqrt{6}$ units.

$\therefore$ the magnitude of the length of AB is the geometrical value of $\sqrt{6}$.

Ex. 4. *A, B, C are three points in order in a straight line. Find a point in the straight line so that PB may be a mean proportional between PA and PC.*



Construction. On AC as diameter describe a semi circle. At B draw **perp.** to AC. Bisect the rt. $\angle ABR$ by BM meeting the semi-circle at M. **From M** draw MP perp. to AC meeting it at P.

Then P shall be the reqd. point.

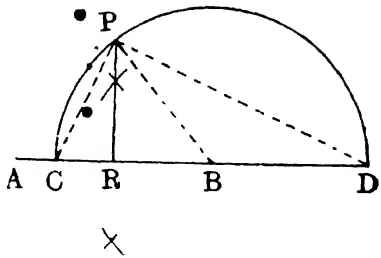
Proof. Now, the $\angle PBM = \angle PMB = 45°$. $\therefore \quad PB = PM$.

Again, the $\angle AMC$ in a semi-circle is a rt. $\angle$, $\therefore \quad PM^2 = PA \,.\, PC$.

$\therefore \quad PB^2 = PA \,.\, PC \qquad [\because \; PM = PB]$

Hence PB is a mean proportional between PA and PC.

Ex. 5. *Having given the difference of the two adjacent sides of a rectang construct it making its area equal to that of a given square.*



Let the difference of the two adjacent sides of the reqd. rectangle be AB and X, a side of the given square.

It is reqd. to describe a rectangle whose area is equal to the square on the side X and the difference of whose adjacent sides is AB.

Construction. Take R the middle pt. of AB at R and draw RP perp. to AB making $RP=X$. Join BP. With centre B and radius BP describe semi-circle cutting BA or BA produced at C, and AB produced at D.

Then shall $CR.RD$ be the reqd. rectangle.

Proof. Join CP, PD.

Now the $\angle CPD$ in a semi circle $=$ a rt. $\angle$ and PQ is perp. to th hypotenuse CD.

$\therefore \quad CR.RD = PR^2 =$ the given square.

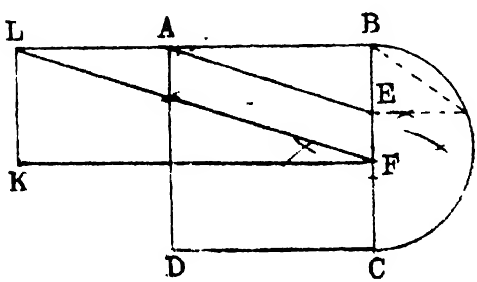
Again, $RD - CR = (BD + BR) - (BC - BR)$

$= 2BR \quad [\because \; BD = BC]$

$= AB$.

Hence the rect. $CR.RD$ is the reqd. rectangle.

Ex. 6. *Construct a rectangle with its sides in a given ratio and area equa to that of a given square.*



Let $ABCD$ be the given square and $m:n$ the given ratio.

It is reqd. to describe a rectangle equal in area to the square $ABCD$ and whose sides will be in the ratio $m:n$.

Construction. Take a pt. E in the side BC of the

are so that $AB:BE=m:n$ and find the mean proportional BF between and BE. Join AE and draw $FL \parallel$ to EA to meet BA produced at L. nplete the rectangle $FBLK$.

Then shall $BLKF$ be the reqd. rectangle.

Proof. $\frac{\text{Square } ABCD}{\text{Rectangle } BLKF}=\frac{BC^2}{BL.BF}=\frac{AB.BC}{BL.BF}$ [$\because AB=BC$]

$=\frac{BE}{BF}\frac{BC}{BF}$ [$\because FL \parallel$ to AE]

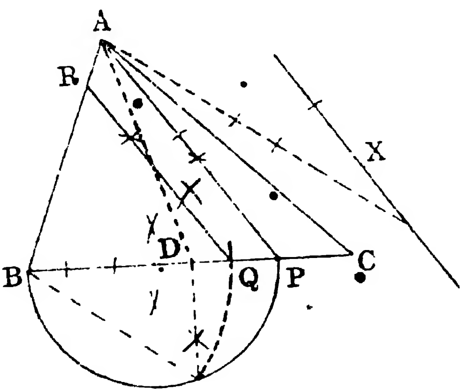
$=\frac{BC.BE}{BF^2}=1$ [$\because BF$ is the mean proportional between BC and BE]

$\therefore$ the rect. $BLKF=$ the square $ABCD$.

Again, because AE is $\parallel$ to FL, $BL:BF=AB:BE=m:n$.

$\therefore$ the ratio of the sides BL, BF is $m:n$.

Ex. 7. *Bisect a triangle by a straight line drawn parallel to a given ight line.*



Let ABC be the given $\triangle$ and X the given str. line.

It is reqd. to bisect the $\triangle ABC$ by a straight line drawn parallel to X.

Construction. Bisect BC at D and through A draw $AP \parallel$ to X to meet at P. Find BQ the mean proportional between BD and BP. Through aw $QR \parallel$ to PA or X to meet AB at R.

Then shall the line QR bisect the $\triangle ABC$.

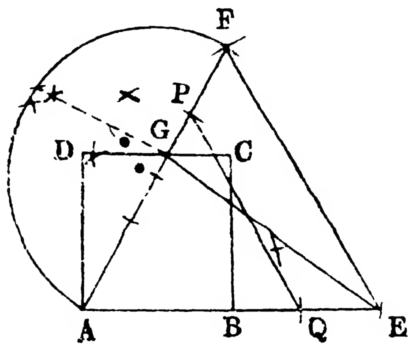
Proof. Join ....

Since the $\triangle^s$ RBQ and ABP are similar,

$$\therefore \frac{\triangle RBQ}{\triangle ABP} = \frac{BQ^2}{BP^2} = \frac{BD \,.\, BP}{BP^2} = \frac{BD}{BP} = \frac{\triangle ABD}{\triangle ABP}.$$

$\therefore$ the $\triangle RBQ$ = the $\triangle ABD = \frac{1}{2}\triangle ABC$, [$\because$ D is the middle pt. of BC.]

Ex. 8. *Construct an equilateral triangle equal in area to a given square.*



Let $ABCD$ be the given square.

It is reqd. to construct an equilateral triangle equal in area to the square $ABCD$.

Construction. Produce AB to E making $BE = AB$. Describe an equilateral $\triangle AFE$ on the line AE. Let the side AF of this $\triangle$ cut CD at the pt. G. Join GE. Find AP the mean proportional between AF and AG. Through P draw PQ $\parallel$ to FE to meet AE at Q.

Then shall the $\triangle APQ$ be the reqd. equilateral $\triangle$.

Proof. Since PQ is $\parallel$ to FE, the $\triangle^s$ APQ, AFE are similar.

$$\therefore \frac{\triangle AFE}{\triangle APQ} = \frac{AF^2}{AP^2} = \frac{AF^2}{AG \,.\, AF} = \frac{AF}{AG}.$$

$$= \frac{\triangle AFE}{\triangle AGE} \quad [\because E \text{ is the vertex of both } \triangle^s]$$

$\therefore$ the $\triangle APQ$ = the $\triangle AGE$.

But the $\triangle AGE$ = the square $ABCD$ [$\because$ they stand between the same parallels and the $\triangle AGE$ is on the base AE which is $2AB$]

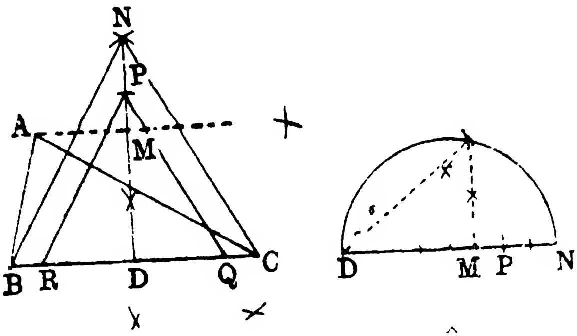
$\therefore$ the $\triangle APQ$ = the sq. $ABCD$.

Ex. 9. *Describe an equilateral triangle equal in area to a given triangle.*

Let ABC be the given triangle.

It is reqd. to describe an equilateral triangle equal in area to the $\triangle ABC$.

Construction. Draw DMN the perp. bis[ector of BC]



Through A draw a str. line ∥ to BC and let it cut DMN at M. Then the $\triangle MBC$ is isosceles and equal in area to the $\triangle ABC$.

With centre C and radius CB describe an arc of a circle to cut DMN at N. Then the $\triangle NBC$ is an equilateral $\triangle$ on the base BC.

Now find the mean proportional DP between DM and DN. Through the pt. P draw PQ ∥ to NC to cut DC at the pt. Q. From DB cut off $DR = DQ$. Join PR.

Then the $\triangle PQR$ shall be the reqd. equilateral $\triangle$.

The proof is left to the students as an exercise.

Ex. 10. *P is a given point within the angle $\angle AOB$. Draw a circle to pass through P, touching OA and OB.*

Let P be the given pt. within the $\angle AOB$.

It is reqd. to draw a circle through P touching OA and OB.

Construction. Bisect the $\angle AOB$ by OX. Draw PN perp. to OX and produce it to meet OA at the pt. R. From NR cut off $NQ = NP$.

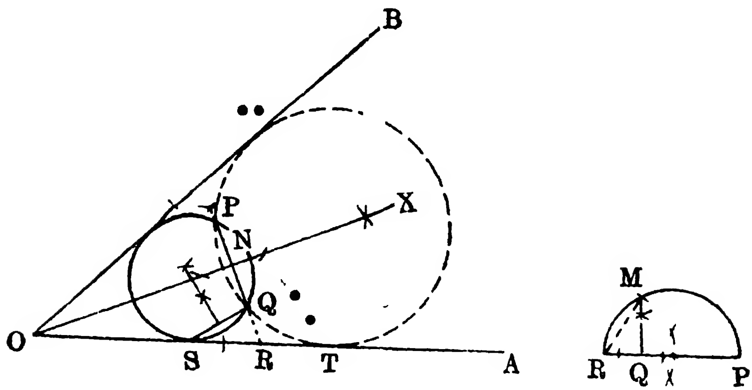
Now, find the mean proportional RM between RP and RQ. On OA cut off from it RS and RT on both sides of R, making each of them equal to RM.

The circle passing through the three pts. S, P, Q will be the reqd. circle.

The circle passing through the three pts. T, P, Q will also be another such circle.

Proof. Since $RS^2 = RM^2 = RP \cdot RQ$.

$\therefore$ the line RS will touch the circle PQS at S.



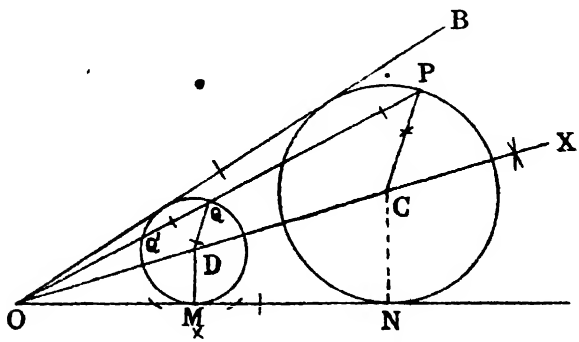
Again, the centre of the circle PQS is on the perpendicular bisector of the chord PQ and since this line bisect the $\angle AOB$, the centre of the circle is equidistant from OA and OB.

$\therefore$ OA being tangent to the circle, OB also must be tangent to the circle.

$\therefore$ the circle PQS through P is one of the reqd. circles.

Similarly, the circle PQT also will be another such circle.

Alternative Method.



Construction. Take any pt. D on the bisector OX of the $\angle AOB$. Draw DM perp. to OA. With centre D and radius equal to DM describe a circle

which touches OA and OD. Join OP cutting this circle at Q. Join QD. Through P draw $PC \parallel$ to QD meeting OX at C. Now, the circle drawn with centre C and radius CP touches OA and OB.

This circle will be the reqd. circle.

Proof. From C draw CN perp. to OA.

Now, the Δ' ODQ, OCP are similar. [$\because DQ$ is $\parallel$ to CP]

$\therefore \quad DQ : CP = OD : OC = DM : CN$. [$\because DM$ is $\parallel$ to CN]

But $DQ = DM$; $\therefore \quad CP = CN$.

Hence, the circle with centre C and radius CP will pass through N and since the $\angle CNA$ is a rt. $\angle$, this circle will touch OA at N.

As the pt. C is on the bisector of the $\angle AOB$, its perp. distances from OA and OB are equal.

$\therefore$ the circle will also touch OB.

N. B. The line OP cuts the circle with centre D at another pt. Q' besides Q. Hence by drawing a str. line through P parallel to $Q'D$ another circle will be obtained.

Exercise IV

1. Divide a right angle into five equal parts.

2. Find geometrically the values of (*i*) $\sqrt{3}$, (*ii*) $\sqrt{5}$ and (*iii*) $\sqrt{14}$.

3. Circumscribe a circle about a regular hexagon.

4. Inscribe a square in a circle of 2 cm. radius and find its side by measurement and calculation.

5. Divide a given straight line into two parts, so that the rectangle contained by the two parts may be equal to a given square.

6. Having given the perimeter, construct a rectangle equal in area to a given (*i*) Rectangle, (*ii*) Square.

7. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

8. Divide a given straight line AB internally at P, such that $AP^2 = 5PB^2$.

[*Hints:* Divide AB in medial section at M. From BA cut off BP equal to $\frac{1}{2}AM$.]

9. Find a point on the circumference of a circle, from which lines drawn to two other given points on the circle will have given ratio.

10. Trisect a given triangle by lines drawn parallel to the base.

11. From the vertex A of a triangle ABC draw a straight line meeting BC produced in D, so that AD may be a mean proportional between the segments of the base.

[*Hints* : Make the $\angle CAD =$ the $\angle ABC$.]

12. Draw a circle passing through two given points, and touching a given straight line, the given points being on the same side of the line, and the line joining the points being not parallel to the given straight line.

Miscellaneous Exercise

1. ABC is a triangle obtuse-angled at A ; AD is drawn perpendicular to BC. Prove that

$$AB^2 + BC \,.\, CD = AC^2 + BC \,.\, BD.$$

2. ABC is a triangle ; find a point P, within it such that $PA^2 + PB^2 + PC^2$ may be a minimum.

3. ABC is an acute-angled triangle ; AD, BE are perpendiculars to BC and CA, intersecting at O. Prove that

the rect. $AD \,.\, OD =$ the rect. $BD \,.\, DC$.

4. ABC is an isosceles triangle whose vertex is A. If BP is perpendicular to AC meeting it at P, and PN perpendicular to BC meeting it at N, prove that

$$AB^2 = AN^2 + PN^2.$$

5. In a triangle ABC, AD bisects $\angle A$ and meets BC in D. The perpendicular to AD through D meets AB and AC in P and Q. Prove that $AB : AC = BP : CQ$.

6. If a straight line AB is divided internally in medial section at M, then $AB^2 + BM^2 = 3AM^2$.

7. Two circles intersect at A and B, and at A tangents are drawn, one to each circle to meet the circumferences at C and D. Show that AB is a mean proportional between BC and BD.

8. The common chord AB of two intersecting circles is produced to meet their common tangents at C and D; if PQ be a common tangent, prove that

$$CD^2 = PQ^2 + AB^2.$$

9. AB is a chord of a circle, and from a point P on the circumference, PC and PD are drawn parallel to the tangents at A and B respectively, meeting AB at C and D. Prove that

$$AC : BD = PA^2 : PB^2.$$

[*Hints :* $\triangle$*s* PAC, BPD *are similar.*]

10. Tangents PA, PB are drawn to a circle; BD is perpendicular to the diameter AC through A. Prove that PC bisects BD.

11. Two circles touch internally at A, and a chord BC of the bigger circle touches the smaller at T. If AB and AC cut the smaller circle at P and Q respectively, prove that

$$BT : CT = AP : AQ.$$

12. O is a fixed point and OP any line drawn to meet the circumference of a fixed circle in P; if OP be divided at Q in a constant ratio, show that the locus of Q is a circle.

13. Construct an equilateral triangle, having given the sum of a side and altitude.

14. A regular hexagon has each side 2 cms. in length, Construct accurately to the scale a square equal to it in area.

15. Divide the hypotenuse of a right-angled triangle into two parts, such that the difference between the squares on them shall be equal to the square on one of the sides containing the right angle.

16. A square being given, construct a circle passing through one angular point, and touching two sides of the square.

[*Hints : Divide the diagonal into two parts, such that the square on one part is twice that on the other.*]

17. Describe a circle touching a given circle, and also touching a given straight line at a given point on it. How many such circles are possible ?

18. Describe a circle touching a given circle, and passing through two given points outside the circle. How many such circles are possible ?

Higher Secondary Syllabus of Elective Mathematics :

SOLID GEOMETRY
(including Mensuration)

(Course for Class X)

(a) Solid Geometry :

Axiom (i) One and only one plane may be made to pass through any two intersecting lines.

Axiom (ii) Two intersecting planes cut one another in a straight line and in no point outside it.

To prove

1. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

2. All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

3. If wo straight lines are parallel and if one of them is perpendicular to a plane, then he other is also perpendicular to the plane.

Concept of angle between two planes and angle between a straight line and a plane.

Concept of parallelism of planes.

Concept of a line being parallel to a plane.

Concept of skew lines.

(b) Mensuration :

Parallelopipeds, Right Circular cones, Prisms and Pyramids (Expressions without proof, of the surfaces and volumes of the solids),

CONTENTS

IMPORTANT FORMULÆ AND RESULTS

Solid Geometry (Mensuration)

1. *Rectangular parallelopiped* (or cuboid).

If a, b, c be its length, breadth and height

(i) Area of the surface $= 2(bc + ca + ab)$.

(ii) Volume $= abc$.

(iii) Surface area of a cube of side $a = 6a^2$.

(iv) Volume $= a^3$.

2. *Right Pyramid on any regular base*

(i) Slant surface $= \frac{1}{2}$(perimeter of base) $\times$ slant height.

(ii) Volume $= \frac{1}{3}$(area of base) $\times$ height.

3. *Tetrahedron.*

Volume $= \frac{1}{3}$(area of base) $\times$ height.

4. *Right Prism.*

(i) Lateral surface $=$ (perimeter of base) $\times$ height.

(ii) Volume $=$ (area of base) $\times$ height.

5. *Right circular cylinder.*

If r is the radius of the base and h the height of the cylinder,

(i) Area of the curved surface

$$= \text{(circumference of base)} \times \text{height}$$
$$= 2\pi rh.$$

(ii) Area of the whole surface

$$= 2\pi rh + 2\pi r^2 = 2\pi r(h + r).$$

(iii) Volume $=$ (area of base) $\times$ height $= \pi r^2 h$.

6. *Right circular cone.*

If r is the radius of the base, h the height, l the slant side and a the semi-vertical angle of the cone,

(i) Area of curved surface

$$= \tfrac{1}{2}(\text{crcumference of base}) \times \text{slant side}$$
$$= \tfrac{1}{2} \cdot 2\pi r \cdot l = \pi r l$$
$$= \pi r \sqrt{h^2 + r^2} = \pi r^2 \operatorname{cosec} a.$$

(ii) Area of the whole surface $= \pi r(l + r)$.

(iii) Volume $= \tfrac{1}{3}(\text{area of base}) \times \text{height}$

$$= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2 a.$$

7. *Sphere.*

If r be the radius of the sphere,

(i) Area of curved surface $= 4\pi r^2$.

(ii) Volume $= \tfrac{4}{3}\pi r^3$.

SOLID GEOMETRY

(To be taught in Class X)

CHAPTER I

FUNDAMENTAL CONCEPTS AND DEFINITIONS

1·1. We give below definitions and chief characteristics of some fundamental entities used in Solid Geometry.

(i) A **point** has position but no magnitude ; that is, it has neither length nor breadth nor thickness.

(ii) A **line** has length but no breadth and thickness.

(iii) A **surface** has length, breadth but no thickness.

(iv) A **solid** has length, breadth and thickness.

Thus, a brick is a *solid*, one of its six faces is a *surface*, an edge is a *line* and a corner is a *point*.

Each of the three elements (1) length, (2) breadth and (3) thickness of a body is called a *dimension* of the body.

Thus, a point has no (or zero) dimension, a line has one dimension, a surface has two dimensions and a solid has three dimensions.

A solid is bounded by surfaces, a surface is bounded by lines and a line is bounded by points.

Solid Geometry deals with the properties of lines, surfaces and solids in three dimensional space.

1·2. If a surface be such that the straight line joining *any two* points in it lies wholly on the surface, it is called a plane surface or more simply a *plane*.

Note. In this treaties straight lines are supposed to be of infinite length and the planes of infinite extent unless anything to the contrary is stated. The expression *lies wholly on the surface* means that every point in the straight line however produced both ways lies in the surface.

1·3. Lines are said to be **co-planar** if they be in a plane or a plane can be made to pass through them.

1·4. Two straight lines are said to be *parallel*, when they *lying in the same plane*, do not meet however far they may be produced both ways.

It should be noted that every pair of parallel straight lines is co-planar.

1·5. Lines are said to be **skew** when a plane cannot be made to pass through them and they do not meet however far they may be produced.

Thus, it should be noted that skew lines neither intersect nor are they parallel.

From (1·3), (1·4), (1·5), it is clear that two straight lines are either co-planar or skew.

If they are co-planar, they either intersect or are parallel and if they are not co-planar *i.e.*, if they are skew, they neither intersect nor are they parallel.

1·6. **Planes** are said to be **parallel** when they do not meet even if they are indefinitely produced in any direction.

1·7. A **straight line** and a **plane** are said to be **parallel** when they do not meet even if produced indefinitely in any direction.

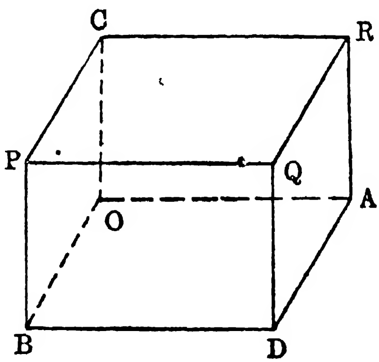


Fig. 1

In the above figure, the lines *PQ*, *BD* are parallel; they lie in the plane *PQDB*. *PQ*, *OB* are skew lines. Planes *PQDB*, *OARC* are parallel. Straight line *PQ* is parallel to the plane *OARC*.

1·8. A straight line is said to be **perpendicular** to a plane, if it is perpendicular to every straight line which meets it in that plane.

Such a straight line is said to be **normal** to that plane.

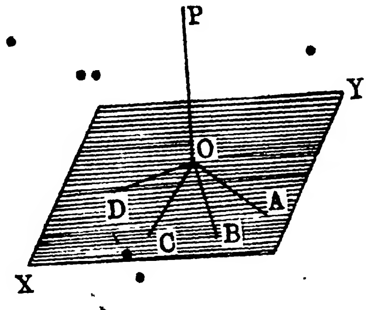


Fig. 2

Thus, PO is said to be perpendicular to the plane XY if it is perpendicular to every straight line OA, OB, OC, OD etc. drawn in the plane to meet it at O.

1·9. If a straight line is parallel to the direction of a plumb-line hanging freely at rest, it is called a **vertical line.** The plane which is perpendicular to the vertical line is called the **horizontal plane.**

Any straight line drawn in the horizontal plane is called a *horizontal* line.

1·10. The *angle between two skew straight lines* (*i.e.*, two non-co-planar straight lines) is measured by the angle contained by one of them and a straight line drawn through any point in that line parallel to the other.

Let AB and CD (Fig. 3) be two skew straight lines; through any point P on AB draw the straight line PQ parallel to CD. Then $\angle QPB$ is the angle between the skew st. lines AB and CD.

Since, the sides of a triangle all lie in one plane, a triangle is a plane figure; a parallelogram is also a plane figure, but all the sides of a quadrilateral need not lie in one plane, so a quadrilateral may or may not be a plane figure. If a quadrilateral be drawn such that two of its adjacent sides lie in one plane and the other two in another plane, such a quadrilateral is called

skew or gauche. If the extremities of a pair of finite skew lines be joined, a skew quadrilateral is formed.

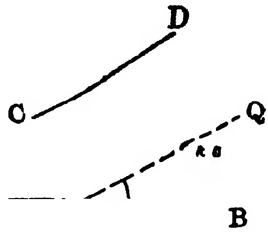


Fig. 3

That the sides of a quadrilateral need not lie in one plane can be easily seen by folding a plane quadrilateral about either diagonal.

1·11. The locus of the feet of the perpendiculars drawn from all points in a line on a given plane is called the *orthogonal projection* (or simply *the projection*) of the line on the plane.

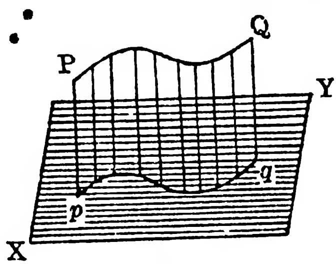


Fig. 4

In the above figure (4), the projection of the line PQ on the plane XY is the line pq.

The projection of a straight line is itself a straight line. A straight line and its projection are co-planar.

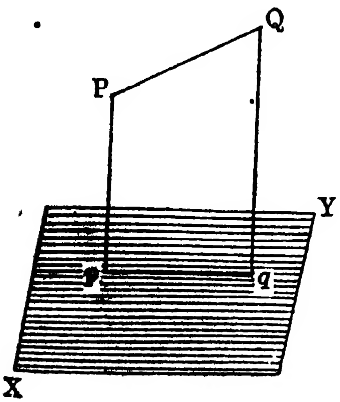


Fig. 5

In the adjoining figure (5), the straight line pq is the projection of the straight line PQ. PQ, pq are co-planar.

The projection of a curved line on a plane may be a curved line, as in Fig. 4, or it may sometimes be a straight line

when the curved line lies on a plane perpendicular to the plane of projection.

1·12. The *angle between a straight line and a plane* is measured by the angle between the straight line and its projection on the plane.

Let the straight line PQ (Fig. 6) and its projection pq (on the plane XY) produced if necessary meet at the point R in the plane XY (as shown in the adjoining figure).

Then $\angle QRq$ is called the angle between the straight line PQ and the plane XY.

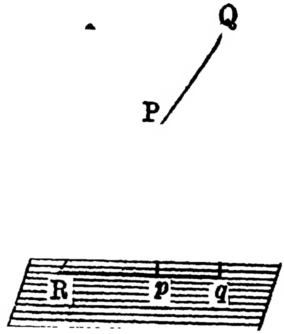


Fig. 6

1·13. The *angle between two planes* is measured by the plane angle contained by the two straight lines drawn from *any* point in the line of section of the two planes, perpendicular to that line of section, one in each plane.

This angle is called a **dihedral angle** between the two planes.

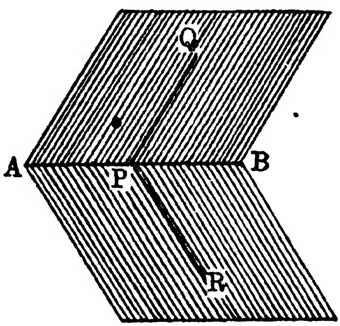


Fig. 7

Let π_1, π_2 be two planes and let AB be their line of intersection. From *any* point P in AB, draw two straight lines PQ, PR both perpendicular to AB, one in each plane. Then the angle between the two planes is measured

CHAPTER II

AXIOMS AND THEOREMS

Axioms.—The following fundamental properties have been laid down as axioms.

(1) One and only one plane may be made to pass through any two intersecting straight lines.

(2) Two intersecting planes cut one another in a straight line and in no point outside it.

From above, the following inferences can be drawn at once :

The position of a plane is fixed if it passes through

(1) a given straight line and a point outside it.

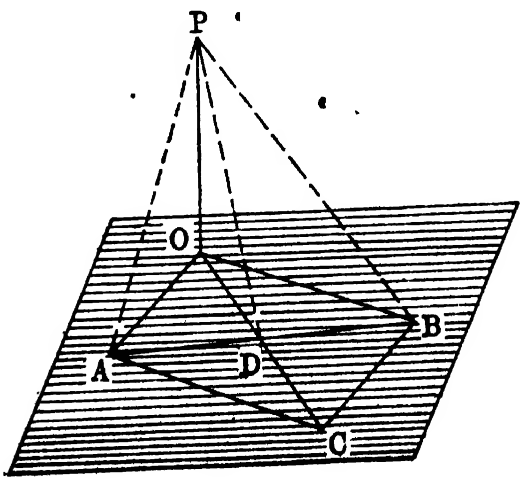
(2) two intersecting straight lines.

(3) three non-collinear points.

(4) two parallel straight lines.

THEOREM I

If a straight line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane in which they lie.



Let OP be perpendicular to each of two intersecting lines OA, OB at their point of intersection at O.

It is required to prove that PO is perpendicular to the plane AOB.

Through O draw *any straight line* OC in the plane AOB. Also in the same plane, through C, draw CA parallel to OB, and CB parallel to OA, so that $OACB$ is a parallelogram. Join AB to meet OC in D, so that D is the mid-point of AB.

[$\because$ *diagonals of a parallelogram bisect each other*]

Join PA, PB, PD.

Proof. In the triangle PAB, the base AB is bisected at D.

$$\therefore \quad PA^2 + PB^2 = 2AD^2 + 2PD^2 \quad \cdots \quad \cdots \quad (1)$$

Similarly from the triangle OAB, we get

$$OA^2 + OB^2 = 2AD^2 + 2OD^2 \quad \cdots \quad \cdots \quad (2)$$

$\therefore$ subtracting (2) from (1),

$$(PA^2 - OA^2) + (PB^2 - OB^2) = 2(PD^2 - OD^2) \quad (3)$$

$$\left.\begin{aligned} \text{Now, } PA^2 - OA^2 &= OP^2 ; \\ \text{and } \quad PB^2 - OB^2 &= OP^2 \end{aligned}\right\} \quad \cdots \quad \cdots \quad (4)$$

since, $\angle POA$, $\angle POB$ are right angles.

$\therefore$ from (3) and (4), we ge

$$2OP^2 = 2(PD^2 - OD^2).$$

$$\therefore \quad OP^2 + OD^2 = PD^2$$

$\therefore$ $\angle POD$ *i.e.*, $\angle POC$ is a right angle.

Thus, PO is perpendicular to *any line* OD which meets it in the plane AOB.

$\therefore$ PO is perpendicular to the plane AOB. **Q. E. D.**

Alternative method of Proof.

Let PO be perpendicular to both OA, OB.

It is required to prove that PO is perpendicular to the plane of OA, OB *i.e.* the plane AOB.

Join AB and in the plane of AOB draw *any straight line* OC, meeting AB in C. Then it will be sufficient to prove that PO is perpendicular to OC. Produce PO beyond the plane OAB to Q, making $OQ = OP$.

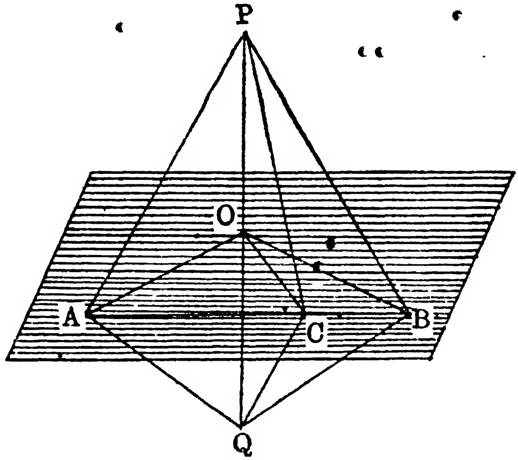


Fig. 9

Join PA, PB, PC ; also QA, QB, QC.

Now AO being perpendicular bisector of PQ, $PA = QA$ and BO being perpendicular bisector of PQ, $PB = QB$.

Hence, $\Delta^s PAB$, QAB are congruent (AB being common to both). $\therefore$ $\angle PAB = QAB$.

Then in Δ^s PAC, QAC, $PA = QA$, AC is common and $\angle PAC = \angle QAC$. $\therefore$ Δ^s PAC, QAC are congruent. $\therefore$ $CP = CA$.

Now in the Δ^s OPC & OQC, $OP = OQ$, $PC = QC$ and OC is common. $\therefore$ Δ^s OPC, OQC are congruent.

$\therefore$ $\angle POC = \angle QOC$ which being adjacent angles on the straight line POQ, $\angle POC = \angle QOC =$ a right angle.

$\therefore$ PO is perpendicular to *any line* OC which meets it in the plane AOB.

$\therefore$ PO is perpendicular to the plane of OA, OB. Q. E. D.

Exercise 1

1. Show that from a point in space, three straight lines can be drawn so that each is perpendicular to the plane of the other two.

2. A straight line is drawn through the centre O of a circle perpendicular to the two radii OA, OB of the circle. Show that all points on the circumference of the circle are equidistant from any point on the line.

[Let P be any point on the line OL, drawn perp. to OA, OB. Hence PO is perp. to the plane of the circle. Let C be any point on the circumference, Then PO is perp. to OC. Then, since $OA = OC$, PO is common and $\angle POA = \angle POC$, each being a right angle. $\therefore$ Δ' POA, POC are identically equal. $\therefore$ $CP = AP$. Since P and C are any points, theorem is proved.]

3. If O be a point in the plane of the triangle ABC and if P be a point outside the plane such that PO is perpendicular to OA and OB and if $PA = PB = PC$, show that O is the circum-centre of the triangle ABC.

[Since PO is $\perp$ to OA, OB, PO is perp. to the plane of the triangle ABC and hence $\perp$ to OC. In the Δ' POA, POB, since $PA = PB$, $\angle POA = \angle POB$ each being a right angle, and PO is common, Δ' POA, POB are identically equal. $\therefore$ $OA = OB$. Similarly from Δ' POA, POC, $OA = OC$. $\therefore$ $OA = OB = OC$.]

4. If O be the circum-centre of any given triangle ABC and if P be any point outside the plane of the triangle ABC such that $PA = PB = PC$, show that PO is perpendicular to the plane ABC.

5. P is any point outside a given plane, and O, A, B, C, D are points in the plane such that $\angle POA = \angle POB =$ a right-angle. If $PA = PB = PC = PD$, prove that the points A, B, C, D are concyclic. Find the centre of the circle, passing through A, B, C, D.

6. ABC is a triangle right-angled at C. P is a point outside the plane ABC, such that $PA = PB = PC$. If D be the mid-point of AB, prove that PD is perpendicular to CD and hence deduce that PD is perpendicular to the plane of the triangle.

[Here $AD = BD = CD$. $\therefore$ Δ' PDA, PDB, PDC are identically equal. $\therefore$ $\angle PDA = \angle PDB = \angle PDC$: Now since $\angle$' PDA, PDB, are adjacent, each is a right angle. $\therefore$ $\angle PDC$ is a right angle. Hence PD is perp. to DA, DC.]

THEOREM II

All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

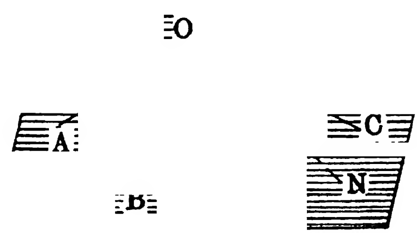


Fig. 10

Let PO be the given straight line and let the straight lines OA, OB, OC be drawn perpendicular to PO at O.

It is required to prove that OA, OB, OC are co-planar *i.e.*, they all lie in one plane.

Proof. Evidently OA, OB lie in one plane and PO, OC in another plane.

If possible, let the plane POC cut the plane AOB in the straight line ON.

Now, since PO is perpendicular to OA and OB, PO is perpendicular to the plane AOB.

But ON being in the plane AOB and meeting PO at O, PO is perpendicular to ON; hence $\angle PON$ is a right angle. Also $\angle POC$ is a right angle.

Now, PO, OC, ON lying in the same plane, $\angle POC$, $\angle PON$ cannot be both right angles. Hence, ON coincides with OC. Thus, OC is the line of intersection of the planes AOB and POC. Hence OC lies in the plane of OA, OB.

Similarly it can be easily shown that if other lines OD, OE, OF etc. are drawn perpendicular to PO at O, they lie in the plane of AOB. Hence, OA, OB, OC, OD, OE, OF etc. are all co-planar.

Q. E. D.

Exercise 2

1. How many horizontal straight lines can be drawn through a given point of a vertical line and how do they lie?

How many vertical lines can be drawn through a given point?

[An infinite number of horizontal lines; they all lie in one plane. One vertical line.]

2. Through the mid-point O (*i.e.*, the intersection of its diagonals) of a horizontal square $ABCD$, a vertical line OP is drawn. Show that PA, PB, PC, PD are all equal.

[Horizontal square means a square lying on a horizontal plane.

OP being perp. to the plane $ABCD$, is perp. to the diagonals AOC, BOD at their common mid-point O. Hence $\triangle$' POA, POB, POC, POD are congruent.]

3. Prove that there cannot be more than three mutually perpendicular straight lines in space meeting at a point.

4. If a triangle revolves about its base, show that the vertex describes a circle.

[Let AM be drawn perp. from the vertex A to the base BC, (or BC produced). Let $AM=p$; since p is constant, vertex will describe a circle of radius p and centre M. The different positions of the line AM are co-planar each being perpendicular to the base BC.]

5. Find the locus of a point in space—

(i) equidistant from two given points;

[Let P and Q be two points. P and Q is joined and PQ is bisected at O. A plane is drawn through O perpendicular to PQ. Then this plane is the required locus.

Let A be any point in this plane. AO, AP and AQ are joined.

Since OA is a straight line lying in this plane and OQ is perpendicular to this plane, OQ is perpendicular to OA.

Now considering $\triangle AOP$ and $\triangle AOQ$, $OP=OQ$; OA is common and

Similarly it can be proved any point in this plane is equidistant from P and Q. Therefore this point is the required locus.]

(ii) equidistant from three given non-collinear points.

[Let A, B, C be three non-collinear points. Points in space equidistant from A and B lie on a plane perpendicular to AB and passing through mid-point of AB [as in Ex. 5(*i*)]. Similarly points in space equidistant from B and C lie on a plane passing through mid-point of BC and perpendicular to it. These planes intersect each other along a line. Points upon these straight lines being common to both planes equidistant from A, B, C. Therefore the locus is the line of intersection of these two planes.]

6. Find a point in a given straight line in space which is equidistant from two given points outside the line. When is this impossible ?

[Let C and D be two points outside the given straight line. Consider a plane passing through the mid-point of CD say O and perpendicular to CD. Let this plane intersect AB at P.

Therefore $PC^2 = PO^2 + OC^2$ and $PD^2 = PO^2 + OD^2$.

$\therefore$ $PC = PD$ since $OC = OD$.

Hence P is the required point.

If CD is perpendicular to AB, then the plane passing through O perpendicular to CD becomes parallel to AB. Therefore to find out a point P becomes impossible.]

7. Prove that a point can be found in a plane equidistant from three points outside the plane. State the exceptional case, if any.

[Let p be the given plane and A, B, C be three points outside this plane. Consider a plane p_1 which passes through the mid-point of AB and perpendicular to AB. Then any point on the plane p_1 is equidistant from A and B. Similarly let p_2 be the plane which passes through the mid-point of BC and perpendicular to BC. Then any point on the plane p_2 is equidistant from B and C. Therefore every point on the line of intersection of the two planes p_1 and p_2 is equidistant from A, B and C. Let this line of intersection intersect the plane p at the point X. Then X is equidistant from A, B, C.

Therefore X is the required point. If the line of intersection of the planes p_1 and p_2 be parallel to the plane p, no such point can be found.]

8. Show that there is one and only one point equidistant

[Let A, B, C, D be four points not lying in one plane and no three of which are collinear.

All points in space equidistant from A and B must lie on a plane which passes through the mid-point of AB and perpendicular to AB. Similarly the points equidistant from B and C and those from C and D respectively lie on two planes passing through the middle points BC and CD and perpendicular to BC and CD respectively. These three planes mutually intersect in three straight lines. Thus three straight lines meet at only one point common to three planes and this point is obviously equidistant from A, B, C, D.

Hence there is only one point equidistant from A, B, C, D.]

THEOREM III

If two straight lines are parallel and if one of them is perpendicular to a plane, then the other is also perpendicular to the plane.

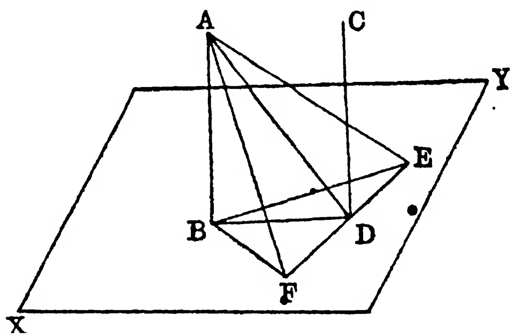


Fig. 11

Let AB, CD be two parallel straight lines of which AB is perpendicular to the plane XY.

It is required to prove that CD is also perpendicular to the plane XY.

Let B, D be the points at which the given lines meet the plane. Join AD, BD.

In the plane XY, draw the straight line EDF perpendicular to BD, making $DF = DE$.

Join AE, AF ; BE, BF.

Proof. Since BD bisects EF at right angles.

$\therefore \quad BE = BF.$

Now in the Δ^s ABE, ABF, AB is common and $BE = BF$; also $\angle ABE = \angle ABF$, since each is a right angle because AB is perpendicular to the plane XY and BE, BF lie in the plane.

$\therefore \quad \Delta^s$ ABE, ABF are congruent.

Hence, $AE = AF$.

Again in the Δ^s ADE, ADF, AD is common, $DE = DF$ and $AE = AF$.

$\therefore \quad \Delta^s$ ADE, ADF are congruent.

$\therefore \quad \angle ADE = \angle ADF$ and these being adjacent angles on the straight line FDE, FD is perpendicular to AD.

Since FD is perpendicular to DA and DB, FD is perpendicular to the plane DAB.

As AD, DB both lie in the plane of the parallel lines AB, CD, four lines AB, CD, AD, BD are co-planar. Hence, CD lies in the plane of ABD and thus FD is perpendicular to CD.

Since AB, CD are parallel and BD meets them,

$\therefore \quad \angle CDB + \angle ABD = 2$ right angles.

But $\angle ABD =$ a right angle.

$\therefore \quad \angle CDB$ is a right angle.

Hence, CD is perpendicular to BD.

Thus, CD being perpendicular to both DB, DF, is perpendicular to the plane XY in which they lie.

$\therefore \quad CD$ is perpendicular to the plane XY. Q. E. D.

Cor. 1. It can be easily proved that the *converse of the theorem is true*; *i.e., if two straight lines are both perpendicular to a plane, then they are parallel.*

Let AB, CD be both perpendicular to the plane XY. Then with the same construction, it can be proved, as before, that FD is perpendicular to DA. But FD is perpendicular to DB (*construction*) and is also perpendicular to DC, since FD lies in the plane XY and CD is perpendicular to the plane XY (*Hyp.*) Hence, DC lies in the plane of DB, DA [*Theo. II*]. But AB also lies in the same plane and hence AB, CD are co-planar and since $\angle ABD = \angle CDB$, each being a right angle, $\angle ABD + \angle CDB = 2$ right angles. Hence, AB is parallel to CD.

Cor. 2. **Theorem of three perpendiculars.**

If AB is perpendicular to a plane XY and if from B, the foot of the perpendicular, a line BC is drawn perpendicular to any straight line DE in the plane, then AC is also perpendicular to DE.

Proof. Through B draw FG *in the plane* XY, parallel to DE. Since BC is perp. to DE, it is also perp. to FG. Again, AB being perp. to the plane XY, AB is perp. to FG. Thus FG being perp. to AB, BC, is perp. to the plane ABC and hence DE being

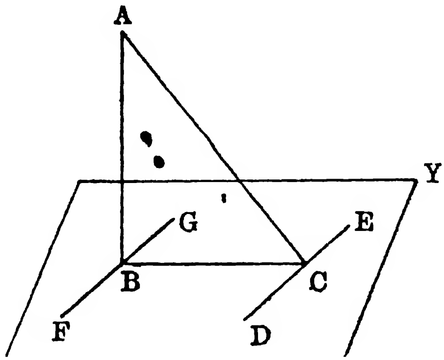


Fig. 12

parallel to FG is also perp. to the plane ABC and hence to the line CA.

Thus, AC is perpendicular to DE.

Exercise 3

1. If the straight lines AB, CD are perpendicular in a plane meeting it at B, D and if AB, CD are equal in length and on the same side of the plane, show that $ABCD$ is a rectangle.

2. *Straight lines in space which are parallel to a given straight line are parallel to one another.*

[Let the straight lines AB, CD be each parallel to the line EF. Draw any plane PRQ perpendicular to EF meeting AB and CD at P and Q and EF at R. Then since AB is || to EF and EF is perpendicular to the plane

PRQ, $\therefore$ *AB* is also perpendicular to the plane *PRQ*. Similarly *CD* is perpendicular to the plane *PRQ*. $\therefore$ *AB* and *CD* which are both perpendicular to the same plane *PRQ*, are parallel (by Cor. 1).]

3. If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed lies in one plane and form a parallelogram.

[Let *P*, *Q*, *R*, *S* be the middle points of the sides *AB*, *BC*, *CD*, *DA* of a skew quadrilateral. *PQ*, *QR*, *RS*, *SA* and *BD* are joined. Since *P* and *S* are mid-points of *AB* and *CD*, *PS* is parallel and half of *BD*. (Note *PS* and *BD* lie in the same plane *BAD*.) Similarly *QR* is parallel and half of *BD*. $\therefore$ *PS* and *QR* are parallel. Hence there is a plane passing through *PS* and *QR*.

Similarly *PQ* is parallel to *RS* and therefore there is a plane which passes through *PQ* and *RS*. But the two different planes cannot pass through four points *P*, *Q*, *R*, *S* at a time. Hence two planes must be same. In a plane, *PS* is equal and parallel to *QR*, hence *PQRS* is a parallelogram.]

4. *P* is a point outside the plane of two parallel straight lines *AB*, *CD*. From the point *P*, *PL* is drawn perpendicular to *AB* and *LM* is drawn perpendicular to *CD*. Prove that *PM* is perpendicular to *CD*.

[Since *AB* and *CD* are parallel, therefore *LM* is parpendicular to both *AB* and *CD*. Consider a plane through *LM* and *P*. Since this plane is perpendicular to the plane containing *AB* and *CD*, therefore *PM* is perpendicular to *CD*.]

5. If *AB*, *CD*, *EF* are three equal, parallel straight lines not lying in one plane, and if their extremities form two triangles *ACE*, *BDF*, show that the triangles are congruent.

6. If perpendiculars are drawn from a point to a system of parallel straight lines in space, show that they lie on a plane perpendicular to the parallel lines.

[Let *AB*, *CD*, *EF*, ... etc. be a system of parallel straight lines lying in the same plane *p* and *OP*, *OQ*, *OR*, ... be respectively drawn perpendiculars to them from an external point *O*. Let *XY* be drawn parallel to *AB*, *CD*, *EF*, ... through *O*. Since *AB* is parallel to *XY* and *OP* is perpendicular to *AB*, *OP* is perpendicular to *XY*. Similarly *OQ*, *OR*, ... etc. are perpendicular to *XY* at *O*. Therefore the perpendiculars *OP*, *OQ*, *OR*, ... etc. must lie in the same plane and that plane will be perpendicular to *XY* i.e. *AB*, *CD*, *EF*, ... etc.]

CHAPTER III

VOLUMES AND SURFACE AREAS OF REGULAR SOLIDS

3·1. When any portion of space is bounded by one or more surfaces, it is called a *solid figure* or simply a *solid*. These surfaces are called the faces of the solid and the intersections of the adjacent faces are called its *edges*.

When a solid is bounded by plane faces, it is called a *polyhedron*. A polyhedron is said to be *regular*, if its faces are all regular, such as equilateral triangles, squares, etc.

3·2. Parallelopiped.

If a solid is bounded by three pairs of parallel planes, it is called a *parallelopiped*.

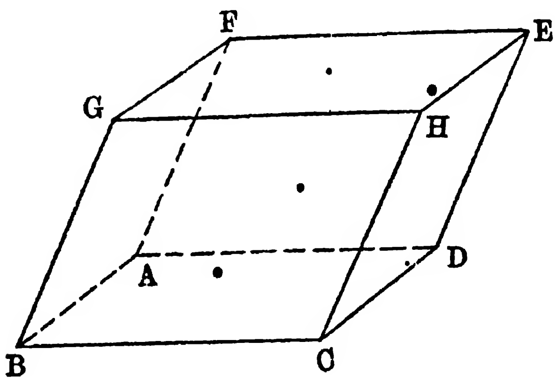


Fig. 13

Here $ABCDEFGH$ is a parallelopiped. Its faces are all parallelograms.

Rectangular parallelopiped.

If a parallelopiped has its faces all rectangles, it is called a *rectangular parallelopiped* (or a cuboid)

Let $OADBPQRC$ be a rectangular parallelopiped. Here OA, OB, OC are three mutually pependicular lines.

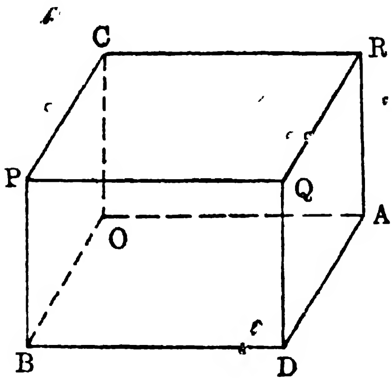


Fig. 14

Here, $\angle COA$, $\angle COB$ are right angles.

$\therefore$ OC is perpendicular to the face $AOBD$.

$\therefore$ OC is perpendicular to OD, as OD lies in the plane $AOBD$.

Hence, QD being parallel to OC is perpendicular to OD.

$\therefore$ $\angle ODQ$ is a right angle.

$\therefore$ from $\triangle OQD$, $OQ^2 = OD^2 + DQ^2 = OD^2 + OC^2$, since $OC = DQ$.

Now $\angle OAD$ being a right angle, $OD^2 = OA^2 + AD^2 = OA^2 + OB^2$ (as $AD = OB$)

$\therefore$ $OQ^2 = OD^2 + OC^2 = OA^2 + OB^2 + OC^2$.

Let $OA = a$, $OB = b$, $OC = c$. $\therefore$ $\mathbf{OQ^2 = a^2 + b^2 + c^2}$.

There are *three* pairs of rectangular faces parallel two by two, *viz.* ($PBDQ$, $COAR$), ($DARQ$, $BOCP$), ($CPQR$, $OBDA$). The opposite faces are congruent. The four diagonals are OQ, AP, BR, CD, and they are all equal.

Whole surface of the rectangular parallelopiped

$$= \mathbf{2(bc + ca + ab)}.$$

Volume $= \mathbf{abc}$

i.e., $=$ *length* $\times$ *breadth* $\times$ *height*

or, $=$ (*area of the base*) $\times$ *height.*

3·3. Cube.

If all the sides of a rectangular parallelopiped are equal (*i.e.*, if the bounding faces are all squares) then the parallelopiped is called a *cube*.

If a denote each side or edge of a cube, then

the whole surface area of the cube $=6a^2$

and the volume $=a^3$ *i.e.*, $=(\text{edge})^3$.

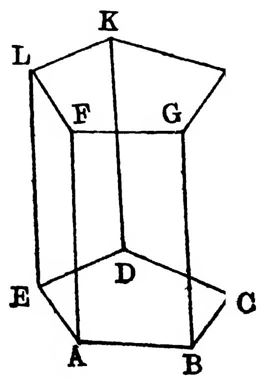
3·4. Prism.

A solid bounded by plane faces of which the side-faces are parallelograms, and the two end-faces called the *ends* are two congruent parallel plane polygons, is called a *prism*.

The straight lines in which the side-faces intersect two by two are called the *side-edges* of the prism. The side-faces being all parallelograms, the side-edges are all parallel and equal and the number of side-faces is equal to the number of the sides of the polygon at the end of the prism. If the two ends of a prism are polygons, it is called a polygonal prism ; *e.g.*, if the two ends are triangles it is called a *triangular prism*.

Right Prism.

A solid bounded by plane faces of which the side-faces are rectangles, and the two end-faces are two congruent parallel plane polygons, is called a *right prism*.



For a right prism, the side-edges are perpendicular to its ends and its height is equal to its edge. The height of a right prism is sometimes called its length. The adjoining figure is that of a right prism.

(1) *Area of the lateral surface of a right prism*

=(perimeter of the base)×height.

(2) *Volume of a right prism*

=(area of the base)×height

3·5. Pyramid.

A solid bounded by plane faces of which one, called the *base*, is any plane polygon, and the remaining faces are all triangles meeting in a point, is called a *pyramid*, the common point of the triangular faces being called its *vertex*. Obviously if the base polygon of a pyramid has n sides, the pyramid has n triangular faces. A pyramid is called a *square pyramid* or *triangular pyramid* according as the base is a *square* or a triangle. The *height* of a pyramid is the perpendicular distance from the vertex to the base. The straight lines in which the triangular faces intersect two by two are called its *edges* (or *slant-edges*).

Right Pyramid.

When a solid is bounded by plane faces of which one called the *base* is a *regular polygon*, and the remaining faces are all isosceles triangles meeting in a point (called the *vertex*), which lies on the straight line drawn perpendicular to the base from its centre (*i.e.*, the centre of the inscribed or circumscribed circle of the polygon), it is called a *right pyramid*.

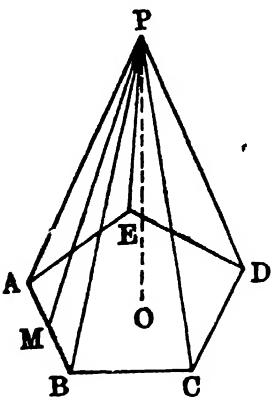


Fig. 16

For a right pyramid the edges of the base are all equal and the side-faces are equal isosceles triangles. The *slant height* of a right pyramid is the length of the perpendicular drawn from the vertex to any side of the base and hence bisecting it. Slant height is the same for each slant face. The *slant surface* of a right pyramid is the sum of its triangular faces. The *height* of a right pyramid is the length of the line joining the vertex to the centre of the base.

The adjoining figure is that of a right pyramid.

The base is $ABCDE$, a regular pentagon here, of which O is the centre. P is the vertex, PO is the height. It has five triangular faces which are all equal. PM is the slant height, bisecting AB at right angles. The above pyramid is very often written as $(P, ABCDE)$

(i) *Slant surface of a right pyramid*

$= \frac{1}{2}$**(perimeter of the base) × slant height.**

(ii) *Volume of a right pyramid*

$= \frac{1}{3}$**(area of the base) × height.**

Note. The *whole* surface = slant surface + area of the base.

3·6. Tetrahedron.

A solid bounded by four triangular faces is called a *tetrahedron*. One of the triangular faces being taken as the base, the point where the other three meet is called the *vertex* of the tetrahedron. If all the four faces of a tetrahedron are equal equilateral triangles, the tetrahedron is said to be *regular*. All the edges of a regular tetrahedron are equal. A tetrahedron is thus a *triangular pyramid*.

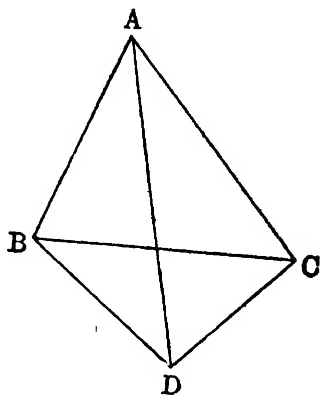


Fig. 17

The adjoining figure is a tetrahedron. It has four triangular faces, ABC, ACD, ADB, BCD and six sides or edges AB, BD, DC, CA, AD, BC. The pairs (AB, CD), (BD, AC), (AD, BC) are called *opposite edges*. A is the vertex and BCD is the *base*. The length of the perpendicular from the vertex A upon the face BCD is called its *height*.

(i) *Whole surface of the tetrahedron*

= sum of the areas of the four faces.

(ii) *Volume of the tetrahedron*

$= \frac{1}{3}$**(area of the base) × height.**

3·7. Right circular cone.

If a solid is generated by the complete revolution of a right-angled triangle about one of the sides containing the right angle as axis, the solid is called a ...

The figure (18) is that of a right circular cone. It is generated by the complete revolution of the right-angled triangle AOB, about the side AO as axis. The hypotenuse AB generates the curved surface of the cone and is called the generating line or the generator of the surface. AB is also called *slant height* of the cone and is usually denoted by l. The other side OB describes the circle $BDCE$ of radius OB and centre O, which is called the base of the cone. The *radius of the base*, OB, is usually denoted by r. The point A is called the *vertex* of the cone and the length of its axis AO is called its *height* and is usually denoted by h. The angle $\angle BAC$ is called the *vertical angle* of the cone, and $\angle OAC$ or $\angle OAB$ is called the semi-vertical angle of the cone.

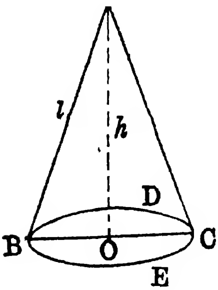


Fig. 18

If h be the height, r the radius of the base, l the slant height and α the semi-vertical angle of the cone,

(i) *Area of the curved surface (or lateral surface) of a right circular cone,*

$$= \tfrac{1}{2}(\text{circumference of the base}) \times \text{slant height}$$

$$= \tfrac{1}{2} \times 2\pi r \times l = \pi r l$$

$$= \pi r \sqrt{h^2 + r^2} = \pi r^2 \operatorname{cosec} \alpha.$$

(ii) *whole surface* $= \pi r\,(l + r)$.

(iii) *volume* $= \tfrac{1}{3}$ (area of the base) $\times$ height

$$= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2\alpha.$$

Note. We easily have from the right-angled triangle AOB

$$r = h \tan \alpha\ ;\ l = r \operatorname{cosec} \alpha.\ l = \sqrt{h^2 + r^2}.$$

Also if v_1, v_2 be the volumes of two right circular cones of heights h_1, h_2 and if they have the same vertical angle 2α,

then, $v_1 : v_2 = \tfrac{1}{3}\pi h_1^3 \tan^2\alpha : \tfrac{1}{3}\pi h_2^3 \tan^2\alpha = h_1^3 : h_2^3$.

[illegible] of cones *with the same vertical angle are to one another*

3·8. Right circular cylinder.

The solid generated by the complete revolution of a rectangle about one of its sides as axis is called a *right circular cylinder.*

The adjoining figure is that of a right circular cylinder. It is generated by the revolution of the rectangle $OABC$ about the side AO. AO is called the *axis* of the cylinder. The opposite side BC generates the curved surface of the cylinder and is called the *generating line* of the cylinder. The side AB describes a *circular end* with A as centre and AB as radius and the opposite side OC describes a *circular end* with O as centre and OC as radius. Both the circular ends are called *bases* and they are equal in area. The length of the axis OA is called the *height* of the cylinder. OA is also sometimes called the length of the cylinder. If r be the radius of the base and h the height of a right circular cylinder, then

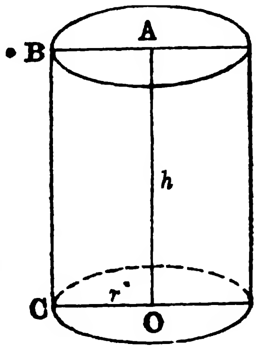


Fig. 19

(i) *area of the curved surface*

$= \textbf{(circumference of the base)} \times \textbf{height}$

$= \mathbf{2\pi rh.}$

(ii) *area of the whole surface*

$=$ area of the curved surface $+$ area of the two ends

$= 2\pi rh + 2\pi r^2$

$= \mathbf{2\pi r(h+r).}$

(iii) *volume* $=$ **(area of the base × height)**

$= \mathbf{\pi r^2 h.}$

3·9. Sphere.

If a solid is generated by the revolution of a semi-circle about its diameter as axis, it is called a *sphere.*

The figure (20) is that of a sphere generated by the complete revolution of the semi-circle APB about the diameter

AB as axis and the semi-circumference APB describes the surface of the sphere. O is the centre of the semi-circle and OP its radius.

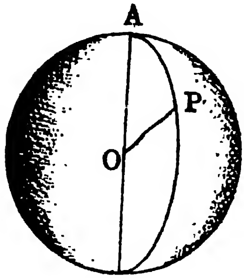


Fig. 20

As the semi-circumference revolves, all the points in it remain at a constant distance from the centre O.

Thus, *a sphere may be defined as the locus of a point which moves in space in such a way that its distance from a fixed point remains constant.*

The fixed point O is called the *centre* of the sphere and the constant distance OP is called its *radius*.

If r be the radius of the sphere,

$$\textit{area of the surface of the sphere} = \mathbf{4\pi r^2}$$
$$\textit{volume of the sphere} = \mathbf{\tfrac{4}{3}\pi r^3}.$$

Note. Half of a sphere is called a hemisphere.

3·10. Illustrative Examples.

Ex. 1. *If the area of the four walls of a room is 1024 sq. ft. and the height is 16 feet, find the perimeter of the floor.*

Let a, b, c be the length, breadth and height of the room. Then $c = 16$ ft.

Then, $2c\,(a+b) = 1024$ sq. ft.,

$$i.e., \quad 2(a+b) = \frac{1024}{c} = \frac{1024}{16} \text{ ft.} = 64.$$

$\therefore$ the perimeter of the floor $= 64$ ft.

Ex. 2. *Find the area of the lateral surface of a right prism whose ends are squares of sides of length 3 inches and whose height is 1 ft.* [C. U.]

Area of the lateral surface $=$ (perimeter of the base) $\times$ height
$= 1$ ft. $\times 1$ ft. $= 1$ sq. ft.

Ex. 3. *Find the volume of the right pyramid in which the base is a triangle whose sides are 8 cm. 15 cm., 17 cm., and the height is 12 cm.*

Since, $8^2+15^2=17^2$, the triangle is a right-angled triangle, of which the hypotenuse is of length 17 c.m.

$\therefore$ area of the triangle$=\frac{1}{2}\times 8\times 15=60$ sq. cm.

Hence, the volume of the pyramid$=\frac{1}{3}$(area of the base)$\times$height

$$=\tfrac{1}{3}\times 60\times 12=240 \text{ cu. cm.}$$

Ex. 4. *Find the whole surface of a regular tetrahedron, the length of each dge of which is 2a.*

Let $ABCD$ be the tetrahedron of which A is the vertex and BCD is the base. Since BCD is an equilateral triangle of side $2a$, its area$=\sqrt{3a\times a\times a\times a}$ [by using the formula for the area of a triangle

$$=\sqrt{s(s-a)(s-b)(s-c)}\ ]=\sqrt{3}a^2.\ ]$$

$\therefore$ whole surface$=$sum of the areas of the four faces, which are here all equal, being equal to the area of the base $BCD=4\times\sqrt{3}.a^2=4a^2\sqrt{3}$.

Ex. 5. *Find the area of the curved surface and the volume of a right circular cone which is 15 ft. high and the radius of whose base is 8 ft.*

[*C. U.*]

Let l be the slant height of the cone ; then $l=\sqrt{15^2+8^2}=17$ ft. Area of the curved surface of the cone$=\pi rl$

$$=3{\cdot}1416\times 8\times 17=427{\cdot}3 \text{ sq. ft. (approx-).}$$

Volume of the cone$=\frac{1}{3}\pi r^2h$, where h is the height of the cone

$$=\tfrac{1}{3}\times 3{\cdot}14\times 8^2\times 15$$

$$=1005{\cdot}31 \text{ cu. ft.}$$

Ex. 6. *The curved surface of a right circular cylinder is 2000 sq. cm. and the diameter of the base is 40 cm. ; find the volume and the height of the cylinder.*

If r be the radius of the base and h the height of the right circular cylinder, area of the curved surface$=2\pi rh=2000$, and $2r=40$.

$$\therefore\ r=20. \qquad \therefore\ \pi h=50 \text{ cm.}$$

Now, volume $V=\pi hr^2=50\times 400=20{,}000$ cu. cm.

Ex. 7. *How many solid spheres, each 6 cm. in diameter, could be moulded from a solid metal right circular cylinder whose height is 45 cm. and diameter 4 cm. ?* [*C. U.*]

Let r_1 be the radius of the sphere ; then its volume$=\frac{4}{3}\pi r_1^{\,3}$.

Here $r_1=3$ cm. $\therefore$ volume of the sphere$=\frac{4}{3}\pi.3^3$.

Let r_2 be the radius and h the height of the cylinder; then its volume $=\pi r_2{}^2 h=\pi\times 2^2\times 45$ (since $r_2=2$, $h=45$).

Let n be the required number of solid spheres moulded.

$\therefore\quad n\times\frac{4}{3}\pi\times 3^3=\pi\times 2^2\times 45.$

$\therefore\quad n=5.$

Thus, 5 spheres can be moulded,

Examples on Chapter III

Sec. A : *(On rectangular parallelopipeds)*

1. The whole surface of a rectangular block is 52 sq. ft. The base contains 6 sq. ft., and one vertical face contains 12 sq. ft. Find the height of the block.

2. The perimeter of the floor of a room is 24 cm., and the total area of the four walls is 144 sq. cm. Find the height of the room.

3. The length, breadth and height of a rectangular block are in the ratio 5 : 6 : 7 ; and the whole surface of the block is 1926 sq. cm. Find the height of the block.

4. The diagonal of a rectangular parallelopiped is 13 cm. The area and perimeter of the floor are 12 sq. cm. and 14 cm. respectively. Find the volume of the parallelopiped.

[Let a, b, c be length, breadth and height respectively.

As in § 3·2, $(13)^2=a^2+b^2+c^2$

Also $12=a.b$

$14=2\,(a+b).$

Solving the last two equations, $a=4$ cm ; $b=3$ cm.

Hence from first equation, $(13)^2=(4)^2+(3)^2+c^2$

$\therefore\quad c=12$ cm.

Volume$\equiv a.\,b.\,c=4.\,3.\,12$ cu. cm $=144$ cu. cm.]

5. The length, breadth and height of a closed box are 10 cm., 9 cm., 7 cm. respectively, and the total inner surface is 262 sq. cm. If the walls of the box are uniformly thick, find the

Sec. B : (*On right Prisms*)

1. The area of the lateral surface of a right prism is 80 sq. in. If the base of the prism be a square of side 4 in., find the height of the prism.

[Perimeter of the base $\equiv 4 \times 4$ in. $= 16$ in.

$\therefore$ $80 = 16 \times h$ [Note § 3·4], where h is the height of the prism.

$\therefore$ $h = 5$ inches.]

2. Show that the volume of a right prism of height h standing on an equilateral triangle of side a is $\dfrac{\sqrt{3}}{4} a^2 h$.

3. The base of a right prism is a trapezium whose parallel sides are 7 cm., and 3 cm., the distance between them being 4 cm. If the volume of the prism be 200 cu. cm., find the height of the prism.

4. Through a wooden pipe, whose cross-section is a square on a side of 4 cm., water flows uniformly at the rate of 50 cm. per sec. How long will it take to discharge 48 litres ?

5. Two prisms of equal height are such that the magnitude of the area of the base of one prism is double the perimeter of the base of the other prism. Show that the magnitude of the volume of the first prism is double the magnitude of the area of the lateral surfaces of the other prism.

[Let $2A$ be the area of base and h height of the first prism and A and h be perimeter and height of second prism.

$\therefore$ Volume of first prism $= 2Ah$.

Lateral surface of second prism $= Ah$.

$\therefore$ Volume of first prism $= 2 \times Ah = 2 \times$ magnitude of area of lateral surface.]

Sec. C : (*On right Pyramids*)

1. A right pyramid of height 1″ stands on a square base of side 4″. Find the slant height and the slant edge.

2. Find the volume of a right pyramid 12 cm. high which stands on a rectangular base of sides 10 cm. and 8 cm.

3. A right pyramid stands on a rectangular base whose sides are 6′ and 8′, and the length of each slant edge is 13′. Find the

[Let $ABCD$ be rectangular base with O as centre and P be the vertex.

Now $BD^2 = BC^2 + CD^2 = (6)^2 + (8)^2 = 100$.

$\therefore$ $BD = 10'$. $\qquad$ $\therefore$ $BO = 5'$.

Again $PB^2 = PO^2 + BO^2$ [PB is slant-edge)

or, $PO^2 = PB^2 - BO^2 = (13)^2 - (5)^2 = 144$.

$\therefore$ $PO = 12$ ft.]

4. Find the volume of the right pyramid whose base is a triangle of which the sides are 13 ft., 14 ft., 15 ft. and whose height is 20 ft.

5. A right pyramid stands on a square base each of whose sides is 12 ft. and the slant faces are equilateral triangles. Find the height and the volume of the pyramid.

6. OA, OB, OC are three mutually perpendicular lines in space, and $OA = a$, $OB = b$, $OC = c$. Prove that the volume of the pyramid $= \frac{1}{6}abc$.

[Area of $\triangle AOB = \frac{1}{2}.\ OA.\ OB = \frac{1}{2}.\ a.\ b.$

Volume of the pyramid $= \frac{1}{3} \times$ area of base $\times$ height

$$= \frac{1}{3} \times \frac{1}{2}.\ ab \times c.$$

$$= \frac{1}{6}.\ abc.\ ]$$

7. OA, OB, OC are three mutually perpendicular lines of equal length a. Find the area of the triangle ABC.

Sec. D : (*On right circular cones*)

1. Find the volume and the area of the slanting surface of a right circular cone of height 4 feet and the radius of whose base is 3 feet ($\pi = \frac{22}{7}$).

2. If S be the area of the curved surface and a the semi-vertical angle, h the height and r the radius of the base of a right circular cone, prove that

$$S = \frac{\pi h^2 \sin a}{2} = \frac{\pi r^2}{\sin a}.$$

[Let a be the semi-vertical angle of this right circular cone.

$\therefore$ $\frac{r}{h} = \tan a$, so that $r = h \tan a$.

$\frac{r}{l} = \sin a$ and $\frac{h}{l} = \cos a$ where l is the slant-height.

Again $S=\pi rl=\pi . h \tan a . \frac{h}{\cos a}=\pi h^2 \frac{\sin a}{\cos^2 a}$;

$$S=\frac{\pi r^2}{\sin a}=\pi h^2 \frac{\sin a}{\cos^2 a}.]$$

3. What is the length of the canvas 3 feet wide which will be required to make a conical tent which is 28 ft. high and which covers an area of 154 sq. yds. ? ($\pi=\frac{22}{7}$)

4. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal curved surfaces. [C. U.]

5. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal volumes.

6. A right circular cone 20 ft. high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis, and if the radius of the base of the original cone be 4 feet, find the volume of the truncated cone. [C. U.]

7. If S_1, S_2 be the curved surfaces and h_1, h_2 the heights of two right circular cones with the same vertical angle, show that

$$S_1 : S_2 = h_1^2 : h_2^2.$$

[Let r_1 and r_2 be the radii of the bases of two cones so that $r_1=h_1 \tan a$ and $r_2=h_2 \tan a$ where a is the semi-vertical angle.

$$S_1=\pi r_1 \sqrt{h_1^2+r_1^2}=\pi h_1 \tan a \sqrt{h_1^2+h_1^2 \tan^2 a}$$

$$S_2=\pi h_2 \sqrt{h_2^2+h_2^2 \tan^2 a}.$$

$$\therefore \quad \frac{S_1}{S_2}=\frac{h_1^2}{h_2^2}.]$$

8. The upper portion of a right circular cone cut off by a plane parallel to the base is removed. If the curved surface of the remainder be $\frac{3}{4}$ of that of the whole cone, show that the cutting plane bisects the altitude of the cone.

[Proceeding as in Ex. 4,

curved surface of $ABC=\pi r \sqrt{h^2+r^2}=\pi h \tan a \sqrt{h^2+h^2 \tan^2 a}$,

curved surface of $AEF=\pi h_1 \tan a \sqrt{h_1^2+h_1^2 \tan^2 a}$.

Now from hypothesis,

curved surface of $EFBC$

$\equiv$curved surface of ABC − curved surface of AEF

$=\frac{3}{4}\times$ curved surface of ABC.

or. $\pi h \tan \alpha \sqrt{h^2+r^2 \tan^2 \alpha} - \pi h_1 \tan \alpha \sqrt{h_1^2 + h_1^2 \tan^2 \alpha}$
$= \frac{3}{4}\pi h \tan \alpha \sqrt{h^2+h^2 \tan^2 \alpha}$,

or, $h^2 - h_1^2 = \frac{3}{4}h^2$,

or, $\frac{1}{4}h^2 = h_1^2$.

$\therefore \quad \frac{h_1}{h} = \frac{1}{2}$.]

Sec. E : (*On right circular cylinders*)

1. If the volume of a right circular cylinder be 1980 cu. ft. and the area of its curved surface be 660 sq. ft., find the radius of the base and the height of the cylinder.

2. If the height of a right circular cylinder be 15·8 ft. and radius of the base be 4·2 ft. find the whole surface of the cylinder. ($\pi = 3·1416$)

[Area of the whole surface $= 2\pi r(h+r)$ [Note § 3·8]
$= 2 \times 3·1416\ (15·8 + 4·2)$ sq. ft.
(since $h = 15·8$ ft., $r = 4·2$ ft.)
$= 527·79$ sq. ft.]

3. Find the height and the volume of the cylinder, the curved surface of which is 2000 sq. ft. and the diameter of whose base is 20 ft.

[Given $\frac{1}{\pi} = 0·3\dot{1}831$]

4. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

5. A right prism stands on a square base whose side is 7·2 cm. Find the volume of a right circular cylinder whose height is 3 cm. and whose base touches the four sides of the square, the centre of the base being at the centre of the square.

[Let $ABCD$ be the square with the centre at O and the base of the right circular cylinder touches the four sides of this square with centre at same point O. Let E be one of the points at which square touches the circle.

$AC^2 = AB^2 + BC^2 = (7·2)^2 + (7·2)^2$.

$\therefore \quad AC = 7·2 \times \sqrt{2}$, $\qquad \therefore \quad AO = OC = 3·6 \times \sqrt{2}$.

Now $OE^2 = OC^2 - CE^2 = (3·6 \times \sqrt{2})^2 - (3·6)^2 = (3·6)^2$.

$\therefore \quad OE = 3·6$ (radius of circular base).

$\therefore$ Volume of cylinder $= \pi r^2 h = \frac{22}{7} \cdot (3·6)^2 \times 3$ cu. ft. $= 122·19$ cu. ft.]

Sec. F : (*On sphere*)

1. Three solid golden balls of radii 3, 4 and 5 millimetres are melted into one single solid golden ball. Find the radius of the single ball. [*C. U.*]

2. A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches. Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere. [*C. U.*]

[Volume of solid sphere $=\frac{4}{3}\pi r^3$.

Volume of right circular cylinder $=\pi r^2 h=\pi r^2 . 16$.

From given condition, $\frac{4}{3}\pi r^3=\pi r^2\, 16$.

$\therefore\ r=12$ inches.]

3. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height ? [*C. U.*]

4. The weights of two balls are in the ratio of 5 to 11 and the weights of a cubic foot of the material in the two balls are in the ratio of 121 to 25. Find the ratio of their radii.

[From given condition

$\dfrac{\frac{4}{3}\pi r^3 . m}{\frac{4}{3}\pi R^3 . m_1}=\frac{5}{11}$ where r and R are radii of two balls and m and m_1 are masses of a cubic foot of two balls.

or, $\dfrac{\frac{4}{3}\pi r^3 . 121}{\frac{4}{3}\pi R^3 . 25}=\frac{5}{11}$ $\qquad\left[\text{since } \dfrac{m}{m_1}=\dfrac{121}{25}\right]$

or, $\left(\dfrac{r}{R}\right)^3=\left(\dfrac{5}{11}\right)^3$. $\qquad\therefore\ \dfrac{r}{R}=\frac{5}{11}$.]

5. If a solid sphere of radius 4 ft. is blown into a hollow sphere, the radius of whose external surface is 5 ft., show that the thickness of the hollow sphere, assuming it to be uniform, is approximately 1·06 ft. [Given $\sqrt[3]{(61)}=3·94$]

6. The volumes of a sphere and of a right circular cylinder are as 4 : 9, and the radius of the base of the cylinder is equal to three times the radius of the sphere. Show that the radius of the sphere is three times the height of the cylinder.

[Let r be radius of the sphere so that $3r$ is the radius of the base of the cylinder.

From given condition,

$$\frac{\frac{4}{3}\pi r^3}{\pi(3r)^2 h} = \frac{4}{9} \text{ where } h \text{ is the height of the cylinder.}$$

or, $\dfrac{\frac{4}{3}r^3}{9r^2 h} = \dfrac{4}{9}$,

$\dfrac{r}{h} = \dfrac{3}{1}$ $\qquad \therefore \quad r = 3h.$]

7. A sphere of diameter 6 cm. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cm. If the sphere be completely submerged, by how much will the surface of the water be raised ?

8. A right circular cylinder is circumscribed about a hemisphere and a right circular cone is inscribed in the cylinder, such that its vertex is at the centre of one end of the cylinder and its base coincides with the other end of the cylinder, show that

$$\frac{\text{Vol. of cone}}{1} = \frac{\text{Vol. of hemisphere}}{2} = \frac{\text{Vol. of cylinder}}{3}.$$

[Volume of the hemisphere $\equiv V_2 = \frac{1}{2}.\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$ where r is the radius of the hemisphere.

Volume of the cylinder $\equiv V_3 = \pi r^2 h$ where r and h are radius and height of the cylinder $= \pi r^3$. [Since cylinder circumscribes hemisphere, $r = h$]

Volume of the cone $\equiv V_1 = \frac{1}{3}\pi r^2 h. = \frac{1}{3}\pi r^3$ [$\because$ $r = h$]

$\therefore \quad \frac{1}{3}\pi r^3 = \dfrac{V_1}{1} = \dfrac{V_2}{2} = \dfrac{V_3}{3}.$]

ANSWERS

Sec. A :—(1) 4 ft. (2) 6 cm. (3) 21 cm. (4) 144 cu. cm. (5) 1 cm.

Sec. B :—(1) 5 inches. (3) 10 cm. (4) 1 minute.

Sec. C :—(1) 2″·236, 3″. (2) 960 cu. cm. (3) 12 ft. (4) 560 cu. ft.

(5) 8·48 ft. ; 407·04 cu. ft. (7) $\frac{\sqrt{3}}{2}.a^2$.

Sec. D :—(1) $37\frac{5}{7}$ cu. ft. ; $47\frac{1}{7}$ sq. ft. (3) 770 ft.

(4) the plane divides the height in the ratio $\sqrt{2}-1 : 1$.

(5) The plane divides the height in the ratio $\sqrt[3]{2}-1$.

(6) $298\frac{1}{3}$ cu. ft.

Sec. E :—(1) radius = 6 ft. ; height = 17·5 ft. (2) 527·79 sq. ft.

(2) 31·8 ft. (5) 122·2 cu. cm.

Sec. F :—(1) 6 m. m. (2) 12 inches. (3) 50%. (4) 5 : 11.

(7) 1 cm.

Higher Secondary Syllabus of Elective Mathematics :

CO-ORDINATE GEOMETRY

(Course for Class X)

Rectangular Cartesian co-ordinates in a plane ; Lengths of segments ; Sections of a finite segment in a given ratio ; Area of a triangle ; Straight line.

CONTENTS

IMPORTANT FORMULÆ & RESULTS

H. S. Co-ordinate Geometry (Class X)

1. Distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   Distance $OP = r = \sqrt{x^2 + y^2}$.

2. Point dividing the line joining two given points in a given ratio : $x = \dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$, $y = \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$.

Middle point $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$.

3. Area of a triangle with given vertices
$$\tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

4. General equation of a straight line
$$ax + by + c = 0 \,(a \text{ and } b \text{ both} \neq 0).$$

Every first degree equation in x, y represents a straight line.

5. Transfer of the origin (directions of axes remaining unchanged) from (0, 0) to (α, β)
$$x = X + \alpha,\ y = Y + \beta.$$

6. Straight line parallel to the x-axis : $y = b$.
   Straight line parallel to the y-axis : $x = a$.

7. Equations of straight lines in standard forms :

(i) Intercept form : $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(ii) 'm' form : $y = mx + c$.

(iii) Form through a given point :
$$y - y_1 = m(x - x_1), \text{ or } \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}.$$

(iv) Normal (or perpendicular) form : $x \cos\alpha + y \sin\alpha = p$.

(v) Two points form : $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

8. Point of Intersection of the two lines

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0:$$

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\quad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

9. Condition for concurrence of the three given lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$: $a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$

10. Condition for collinearity of the three given points (x_1, y_1), (x_2, y_2), (x_3, y_3), is

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

11. Angle between two given lines :

(i) When the lines are $y=m_1x+c_1$, $y=m_2x+c_2$

$$\tan\phi=\frac{m_1\sim m_2}{1+m_1m_2}.$$

(ii) When the lines are

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0$$

$$\tan\phi=\frac{a_1b_2\sim a_2b_1}{a_1a_2+b_1b_2}.$$

12. Conditions for

(a) parallel lines, (i) $m_1=m_2$, (ii) $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$.

(b) perpendicular lines, (i) $m_1m_2=-1$,

(ii) $a_1a_2+b_1b_2=0.$

13. Length of the perpendicular from the point (x_1, y_1) upon the line $ax+by+c=0$ is

$$\pm\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

14. Equations of the bisectors of the angle between the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

CO-ORDINATE GEOMETRY

(To be taught in Class X)

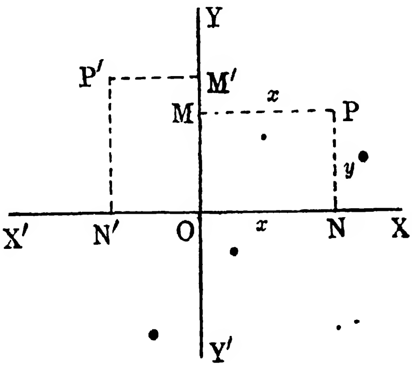
CHAPTER I

RECTANGULAR CARTESIAN CO-ORDINATES; ELEMENTARY RESULTS

1·1. Location of a point on a plane.

Rectangular Cartesian Co-ordinates in a plane.

To locate the position of a point on a plane, we usually take two fixed straight lines on the plane, intersecting one another at right angles, for reference.



These lines are termed the *axes of reference* or *axes of co-ordinates*. One of them, XOX' (from right to left) is usually taken as the *x-axis*, and the other, YOY' (from above downwards) is taken as the *y-axis*. The point of intersection, O, is referred to as the *origin*. The position of a point P on the plane will be definitely known when we know its perpendicular distances PM and PN from the axes of reference. These perpendicular distances, with proper sign are termed the *co-ordinates* or more precisely the rectangular or orthgonal co-ordinates of the point. The distance MP parallel to the x-axis, of P from the y-axis is called the x-co-ordinate, or *abscissa* of P, and conventionally

measured positive from O along OX towards the right, distances from O towards X' being reckoned as negative. Similarly, the distance NP parallel to the y-axis, from the x-axis to the point, is called the y-co-ordinate or the *ordinate* of P, being conventionally measured positive upwards along OY and negative downwards along OY'.

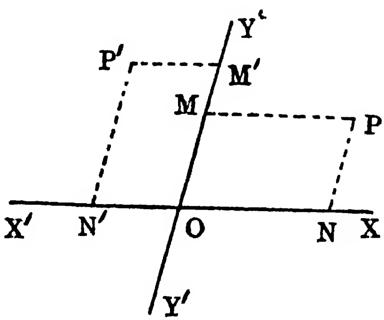
The whole plane is divided by the axes XOX' and YOY' into four quadrants XOY, YOX', $X'OY'$ and $Y'OX'$ and according to the conventions given above, the co-ordinates of a point P are both positive in the first quadrant. In the second quadrant, (for instance for the point P'), the x-co-ordinate is negative and the y-co-ordinate is positive. Both the co-ordinates are negative in the third quadrant, while in the fourth quadrant, the x-co-ordinate is positive but the y-co-ordinate is negative.

Generally, P being any point on the plane of axes of reference XOX' and YOY', PN being drawn perpendicular to the x-axis ON is the x-co-ordinate and NP is the y-co-ordinate of P, and these co-ordinates being given in magnitude and sign, the position of P is definitely fixed. Conversely, if P be given in position on the plane, its co-ordinates are definite in magnitude and sign

The above method of locating a point on a plane is due to Des Cartes, after whom the co-ordinates are referred to as Cartesian Co-ordinates (*Rectangular* or, *Orthogonal*) in this case.

Oblique co-ordinates.

Instead of two mutually perpendicular lines, we may take any two intersecting lines XOX' and YOY' inclined at any angle to one another as

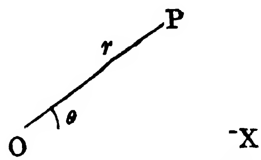


axes of reference and proceed to define the position of a point P, not by its

perpendicular distances from the axes, but by the distances MP and NP of P parallel to $X'OX$ and $Y'OY$ respectively from the other axis. In other words, if PN be drawn parallel to YOY' to meet XOX' at N, ON and NP are respectively the x-co-ordinate and the y-co-ordinate of P, which define the position of P on the plane definitely. The axes are here termed as *oblique axes* and the co-ordinates are *oblique co-ordinates* (Cartesian). The convention as to the signs of the co-ordinates is exactly as in the case of rectangular axes.

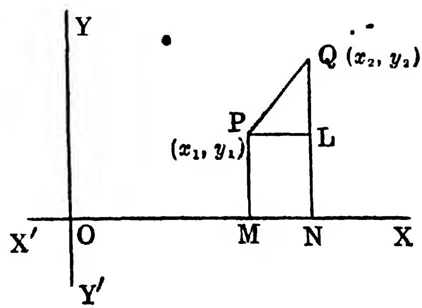
Polar Co-ordinates.

There is yet another method of locating a point on a plane. Here we take a fixed straight line OX on the plane for reference, which we call the *initial line*, and a fixed point O on it, called the *origin* or *pole*. The position of a point P is definitely known if we know the distance OP ($=r$ say), and the angle XOP ($=\theta$ say). The quantities r and θ are called the *polar co-ordinates* of P. The distance r, called the *radius vector*, is always taken as positive, and the angle θ, called the *vectorial angle* is conventionally positive when measured anti-clockwise from OX. The polar co-ordinates being given, the position of P is definite, while if the position of P be given, its polar co-ordinates will have definite values.



Henceforth, throughout the book, we shall use rectangular Cartesian co-ordinates only to refer to positions of points on a plane.

1·2. Lengths of Segments : Distance between two points whose Cartesian co-ordinates are given.



Let P and Q be two given points on a plane whose Cartesian co-ordinates (rectangular) are (x_1, y_1) and (x_2, y_2) respectively.

Let PM and QN be perpendiculars from P and Q on the x-axis OX, and PL perpendicular from P on NQ. Then $OM=x_1$, $MP=y_1$, $ON=x_2$, $NQ=y_2$.

$\therefore \quad PL=MN=ON-OM=x_2-x_1$

$\quad LQ=NQ-NL=NQ-MP=y_2-y_1.$

$\therefore$ from the right-angled triangle PLQ,

$$PQ^2=PL^2+LQ^2=(x_2-x_1)^2+(y_2-y_1)^2.$$

$$\therefore \quad \mathbf{PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.}$$

Cor. The distance r of a point P whose co-ordinates are given to be x, y, from the origin (whose co-ordinates are 0, 0) is given by

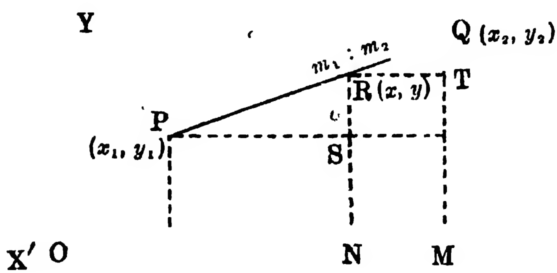
$$OP\equiv \mathbf{r}=\sqrt{\mathbf{x^2+y^2}}.$$

1·3. Section of a given segment in a finite ratio : Point dividing the line joining two given points in a given ratio.

Let P and Q be two given points whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively and let R be the point which divides PQ internally in a given ratio $m_1 : m_2$, *i.e.*, $PR : RQ=m_1 : m_2$.

Let (x, y) be the co-ordinates of R.

Draw PL, QM and RN perpendiculars on the x-axis, and let PS and RT be parallel to the x-axis, meeting RN and QM



at S and T respectively. Then the triangles PSR and RTQ are evidently similar.

Hence, $\dfrac{PS}{RT}=\dfrac{SR}{TQ}=\dfrac{PR}{RQ}=\dfrac{m_1}{m_2}.$

But $PS = ON - OL = x - x_1$, $RT = OM - ON = x_2 - x$.

Similarly, $SR = y - y_1$, $TQ = y_2 - y$.

$$\therefore \quad \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}.$$

Hence, simplifying,

$$\mathbf{x} = \frac{\mathbf{m_1 x_2 + m_2 x_1}}{\mathbf{m_1 + m_2}}, \quad \mathbf{y} = \frac{\mathbf{m_1 y_2 + m_2 y_1}}{\mathbf{m_1 + m_2}},$$

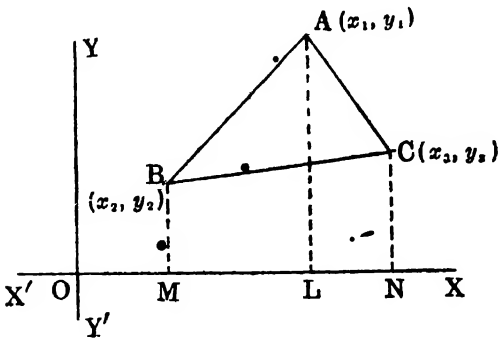
giving the co-ordinates of R.

Cor. The co-ordinates of the **middle point** of the line joining the points (x_1, y_1) and (x_2, y_2) is $\frac{1}{2}(\mathbf{x_1 + x_2})$, $\frac{1}{2}(\mathbf{y_1 + y_2})$.

Note. If R divides PQ *externally* in the ratio $m_1 : m_2$, one of the two quantities m_1 and m_2 is to be taken with a *negative* sign. Whichever of the two is taken negative, the co-ordinates of R will remain the same, namely

$$\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \quad \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}.$$

1·4. Area of a triangle whose vertices are given.



Let the vertices A, B, C of a given triangle have co-ordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

Let AL, BM, CN be drawn perpendiculars on the x-axis, so that $OL = x_1$, $LA = y_1$, $OM = x_2$, $MB = y_2$, $ON = x_3$, $NC = y_3$.

Now evidently,

$$\triangle ABC = \text{trapezium } ABML + \text{trapezium } ALNC - \text{trapezium } BMNC.$$

Also, area of a trapezium $= \frac{1}{2}$ the sum of the parallel sides $\times$ the perpendicular distance between them.

Hence, the area of the triangle ABC is given by

$$\Delta = \tfrac{1}{2}(MB + LA) \times ML + \tfrac{1}{2}(LA + NC) \times LN - \tfrac{1}{2}(MB + NC) \times MN$$

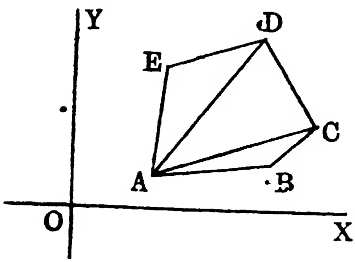
$$= \tfrac{1}{2}(y_2 + y_1)(x_1 - x_2) + \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) - \tfrac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

on simplification.

Note 1. In our figure above, we have taken the vertices A, B, C in such an order that in moving along the sides AB, BC, CA we are to move in an anti-clockwise direction. In such a case, the above expression for the area of the triangle ABC will always be found to be *positive*. If the vertices in order are so taken that in moving along the sides of the triangle in that order we move clockwise, the above expression for the area will be found to be *negative*.

Note 2. Area of a polygon with given vertices.



If a polygon $ABCDE$ has the co-ordinates of its vertices given, to find its area, we break it up into triangles by joining its diagonals, and then add up the areas of these triangles (which have got their vertices known) in the same order. The result will give the required area of the polygon in that order.

1·5. Illustrative Examples.

Ex. 1. *Prove that* $(-1, -1)$, $(1, 1)$ *and* $(-\sqrt{3}, \sqrt{3})$ *are the vertices of an equilateral triangle.*

Let A, B, C be the points whose co-ordinates are $(-1, -1)$, $(1, 1)$ and $(-\sqrt{3}, \sqrt{3})$ respectively.

Then
$$AB^2 = (-1-1)^2 + (-1-1)^2 = 8$$
$$BC^2 = (1+\sqrt{3})^2 + (1-\sqrt{3})^2 = 8$$
$$CA^2 = (-\sqrt{3}+1)^2 + (\sqrt{3}+1)^2 = 8$$

$\therefore \quad AB^2 = BC^2 = CA^2$ or $AB = BC = CA$.

Thus the triangle ABC is equilateral.

Ex. 2. *If the co-ordinates of A, B, C, D are* $(-1, -2)$, $(7, 4)$, $(4, 8)$ *and* $(-4, 2)$ *respectively, show that ABCD is a rectangle.*

Here, $\quad AB^2 = (-1-7)^2 + (-2-4)^2 = 100$

$BC^2 = (7-4)^2 + (4-8)^2 = 25$

$CD^2 = (4+4)^2 + (8-2)^2 = 100$

$DA^2 = (-4+1)^2 + (2+2)^2 = 25$

and $\quad AC^2 = (-1-4)^2 + (-2-8)^2 = 125.$

Thus, $\quad AB = CD$ and $BC = DA$.

Hence, opposite sides being equal, $ABCD$ must be a parallelogram.

Moreover, here, $AC^2 = AB^2 + BC^2$, so that angle ABC is a right angle.

Hence, $ABCD$ is a rectangle.

Ex. 3. *Find the circum-centre and the circum-radius of the triangle whose vertices are the points* $(-4, 3)$, $(-2, -3)$, $(0, -5)$.

Let A, B, C be the vertices of the triangle whose co-ordinates are $(-4, 3)$, $(-2, -3)$, $(0, -5)$ respectively, and let S be the circum-centre of the triangle, whose co-ordinates are supposed to be (x, y).

Then, $\quad SA = SB = SC$ or $SA^2 = SB^2 = SC^2$.

$\therefore \quad (x+4)^2 + (y-3)^2 = (x+2)^2 + (y+3)^2 = (x-0)^2 + (y+5)^2,$

or, $\quad 8x - 6y + 25 = 4x + 6y + 13 = 10y + 25,$

whence solving, $x = 6$, $y = 3$.

$\therefore$ The circum-centre S is the point $(6, 3)$.

Also, circum-radius $= SA = \sqrt{(6+4)^2 + (3-3)^2} = 10$ units.

Ex. 4. *A, B, C, D are four points whose co-ordinates are* $(1, -8)$, $(-3, 4)$, $(0, 7)$ *and* $(3, 16)$ *respectively. Find the ratio in which CD divides AB. Find also the ratio in which AB divides CD.*

Let AB and CD intersect each other at P which divides AB in the ratio $\lambda : 1$.

Then, the co-ordinates of P (by § 1·3) are

$$\frac{-3\lambda+1}{\lambda+1}, \frac{4\lambda-8}{\lambda+1} \text{ respectively.}$$

Now, P, C, D being in one straight line, the area of the triangle PCD is zero. Hence (by § 1·4).

$$\tfrac{1}{2}\left[\frac{-3\lambda+1}{\lambda+1}(7-16)+0\cdot\left(16-\frac{4\lambda-8}{\lambda+1}\right)+3\left(\frac{4\lambda-8}{\lambda+1}-7\right)\right]=0,$$

or, $\quad -9(-3\lambda+1)+3(-3\lambda-15)=0\;;\quad \therefore\ \lambda=3.$

Hence, CD divides AB in the ratio 3 : 1.

The co-ordinates of P are therefore,

$$\frac{3(-3)+1}{3+1},\ \frac{3.4-8}{3+1}\quad i.e.,\ -2,\ 1 \text{ respectively.}$$

Again, if P divides CD in the ratio μ : 1, then co-ordinates of P are $\dfrac{\mu.3+0}{\mu+1}$, $\dfrac{\mu.16+7.1}{\mu+1}$ and these being identical with -2, 1, we have

$$\frac{3\mu}{\mu+1}=-2 \quad\text{and}\quad \frac{16\mu+7}{\mu+1}=1.$$

Either of these give $\mu=-\frac{2}{5}$.

Hence, AB divides CD externally in the ratio 2 : 5.

Ex. 5. *The vertices A, B, C of a triangle have co-ordinates* (5, 6), $(-9, 1)$ *and* $(-3, -1)$ *respectively. Find the length of the perpendicular from A on BC.*

The area of the triangle ABC is

$$\tfrac{1}{2}\,[5(1+1)-9(-1-6)-3(6-1)]=29 \text{ square units.}$$

Also, distance $BC=\sqrt{(-9+3)^2+(1+1)^2}=\sqrt{40}$ units.

Hence, if p be the length of the perpendicular from A on BC, area of the triangle $ABC=\frac{1}{2}.p.\sqrt{40}=29.$

$$\therefore\ p=\frac{29\times 2}{\sqrt{40}}=\frac{29}{\sqrt{10}}=\frac{29}{10}\sqrt{10}\text{ units.}$$

Examples I

1. Find the distance between the following pair of points :

   (i) (2, 1) and (8, 9). (ii) $(-3, 5)$ and $(9, -2)$.

   (iii) $(-6, -7)$ and $(4, -9)$.

   (iv) $(a\cos\alpha,\ a\sin\alpha)$ and $(a\cos\beta,\ a\sin\beta)$.

   (v) $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$.

2. Prove that the point $(-2, -11)$ is equidistant from $(-3, 7)$ and $(4, 6)$.

3. Show that the triangle whose vertices are $(-2, -5)$ $(4, -1)$, $(-1, 0)$ is isosceles.

4. If the ordinate of a point equidistant from $(-3, 7)$ and $(6, -11)$ be 9, find its abscissa.

5. Prove that the points $(4, 8)$, $(4, 12)$ and $(4+2\sqrt{3}, 10)$ are the vertices of an equilateral triangle.

6. Show that the lines joining the point $(4, 5)$ to the points $(-2, -3)$ and $(16, -4)$ are at right angles.

7. Prove that the points $(-2, 3)$, $(-3, 10)$ and $(4, 11)$ are the angular points of an isosceles right-angled triangle.

8. Show that the lines joining successively the points $(-2, -5)$, $(7, -1)$, $(8, 6)$ and $(-1, 2)$ form a parallelogram.

9. Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the successive angular points of a rectangle.

10. Show that (i) the points $(3, -5)$, $(9, 10)$, $(3, 25)$ and $(-3, 10)$ are the vertices of a rhombus;

(ii) the points $(-a, -a)$, $(a, 0)$, $(0, 2a)$, $(-2a, a)$ are the vertices of a square.

11. (i) Prove that the point $(11, 2)$ is the circum-centre of the triangle whose vertices are $(1, 2)$, $(3, -4)$ and $(5, -6)$, and find the circum-radius of the triangle.

(ii) Find the circum-centre of the triangle whose vertices are $(1, 1)$, $(2, 3)$ and $(-2, 2)$.

12. A and B are two fixed points whose co-ordinates are $(2, 4)$ and $(2, 6)$ respectively; ABP is an equilateral triangle on the side of AB opposite to the origin. Find the co-ordinates of P.

[*H. S. 1961*]

13. In a triangle ABC, AD is the median bisecting BC. Prove analytically that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

[*Choose rectangular axes with A as origin.*]

14. G is the centroid of a triangle ABC, and P is any other point on the plane. Prove that

(i) $BC^2 + CA^2 + AB^2 = 3(GA^2 + GB^2 + GC^2)$.

(ii) $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$.

[*Choose rectangular axes with G as origin.*]

15. Find the co-ordinates of the points which divide the line joining the points $(1, -2)$ and $(4, 1)$ in the ratio $2:1$ (i) internally, (ii) externally.

16. The co-ordinates of the points A, B, C, D are $(-2, 3)$, $(1, -2)$, $(8, -3)$ and $(5, 2)$ respectively. Show that AC and BD bisect each other.

17. Find the areas of the triangles whose vertices are respectively,

(i) $(3, -4), (7, 5), (-1, 10)$.

(ii) $(0, 4), (3, 6), (-8, -2)$.

(iii) $(5a, 0), (-2a, 7a), (-6a, -3a)$.

(iv) $(a, b+c), (a, b-c), (-a, c)$.

(v) $(a \cos \alpha, b \sin \alpha), (a \cos \beta, b \sin \beta), (a \cos \gamma, b \sin \gamma)$.

[$\frac{1}{2}\{a \cos \alpha (b \sin \beta - b \sin \gamma) + a \cos \beta (b \sin \gamma - b \sin \alpha)$
$+ a \cos \gamma (b \sin \alpha - b \sin \beta)\}$
$= \frac{1}{2}ab\{\cos \alpha (\sin \beta - \sin \gamma) + \cos \beta (\sin \gamma - \sin \alpha) + \cos \gamma (\sin \alpha - \sin \beta)\}$
$= -\frac{1}{2}ab\{\sin (\alpha - \beta) + \sin (\beta - \gamma) + \sin (\gamma - \alpha)$
$= \frac{1}{2}ab \cdot 4 \sin \frac{1}{2}(\alpha - \beta) \cdot \sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\gamma - \alpha)$
$= 2ab \sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\alpha - \beta)$.]

18. Prove that the area of the triangle whose vertices are the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ respectively is $a^2(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$.

[$\frac{1}{2}\{at_1^2 - (2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_3^2(2at_1 - 2at_2)\}$
$= a^2\{t_1^2(t_2 - t_3) + t_2^2(t_3 - t_1) + t_3^2(t_1 - t_2)\}$
$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$.]

19. Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear. [*The triangle formed by them is of zero area.*]

20. Shew that the points $(0, -3)$, $(3, 0)$ and $(5, 2)$ lie on a straight line.

21. (i) Find the area of the triangle whose vertices are A, B, C are respectively $(3, 4)$, $(-4, 3)$ and $(8, 6)$. Hence, or otherwise find the length of the perpendicular from A on BC. [*H. S. 1961*]

(ii) The vertices A, B, C of a triangle ABC have co-ordinates $(5, 2)$, $(-9, 3)$ and $(-3, -5)$ respectively. Find the length of the perpendicular from A on BC.

22. Find the area of the quadrilateral whose angular points taken in order are (2, 8), (0, −7), (8, −6) and (6, 11).

23. Find the area of the quadrilateral whose angular points are respectively (1, 1), (3, 4), (5, −2), (4, −7). [*C. U. 1944*]

[Let $A(2, 8)$; $B(0, -7)$; $C(8, -6)$ and $D(6, 11)$.

$\therefore$ Area of $\square ABCD =$ Area of $\triangle ABC +$ Area of $\triangle ACD = 9 + \frac{23}{2} = 20\frac{1}{2}$.]

24. (i) Show that the straight line joining the points (0, −1) and (15, 2) divides the line joining the points (−1, 2) and (4, −5) internally in the ratio 2 : 3.

[Let P is the point on which AB intersects CD in the ratio $1 : \lambda$.

$\therefore$ Co-ordinates of P is $\frac{-\lambda+4}{\lambda+1}, \frac{2\lambda-5}{\lambda+1}$ as in § 1·3.

Since P, A, B lie in one straight line, hence area of $\triangle PAB = 0$,

i.e., $\frac{1}{2}\left\{\frac{-\lambda+4}{\lambda+1}(-1-2)+15\left(\frac{2\lambda-5}{\lambda+1}\right)+17\right\}=0,$

whence $\lambda = \frac{3}{2}$. Hence the result.]

(ii) If A, B, C, D are points whose co-ordinates are (−2, 3), (8, 9), (0, 4) and (3, 0) respectively, and AB and CD are joined ; find the ratio of the segments into which AB is divided by CD.

[As above co-ordinates of P is $\frac{8\lambda-2}{\lambda+1}, \frac{9\lambda+3}{\lambda+1}$.

Since P, A, B lie in one straight line, whence area of $\triangle ABC = 0$,

i.e., $\frac{1}{2}\left\{\frac{8\lambda-2}{\lambda+1}(-4)+3\left(4-\frac{9\lambda+3}{\lambda+1}\right)\right\}=0,$

whence $\lambda = \frac{11}{47}$. Hence the result.]

ANSWERS

1. (i) 10. (ii) $\sqrt{193}$. (iii) $2\sqrt{26}$. (iv) $2a \sin \frac{1}{2}(\alpha \sim \beta)$.

(v) $a(t_1 \sim t_2)\sqrt{(t_1+t_2)+4}$. **4.** 23·5. **11.** (i) 10.

(ii) $-\frac{1}{14}, \frac{32}{14}$. **12.** $2+\sqrt{3}$, 5. **15.** (i) 3, 0. (ii) 7, 4.

17. (i) 46 sq. units. (ii) 1 sq. unit. (iii) $49a^2$. (iv) $2ac$.

(v) $2ab \sin \frac{1}{2}(\beta-\gamma) \sin \frac{1}{2}(\gamma-\alpha) \sin \frac{1}{2}(\alpha-\beta)$.

21. (i) $4\frac{1}{2}$ sq. units ; $\frac{3}{17}\sqrt{17}$. (ii) 10·6. **22.** 96 sq. units.

23. $20\frac{1}{2}$ sq. units. **24.** (ii) 11 : 47.

CHAPTER II

EQUATION AND LOCUS

2·1. Equation and Locus.

If an equation in x and y be given, the co-ordinates of any and every point on a plane will not generally satisfy the equation. On the other hand, there are innumerable points on the plane whose co-ordinates will satisfy the given equation ; for we may choose any value of x, and substituting that value in the given equation, solve for value or values of y whereby the given equation is satisfied. We thus get definite point or points with given x, whose co-ordinates satisfy the equation. As x may be chosen arbitrarily, the number of points will be infinite. These points joined will give us a line (straight or curved) on the plane, the co-ordinates of every point of which, (and of no point outside it), will satisfy the given equation. This line is defined as the *locus* represented by the given equation.

Conversely, if a line, straight or curved, be given on a plane then if with chosen axes on the plane, a relation between x and y can be found which is satisfied by the co-ordinates of every point of the given line, and which holds for no other points except those lying on the line, then that relation between x and y is defined as the *equation* of the given line on the plane, with the chosen axes of reference.

For example, given the equation

$$x^2+y^2+6x-4y-12=0,$$

we can write it in the form

$$(x+3)^2+(y-2)^2=25,$$

$$\text{or,}\quad \sqrt{\{x-(-3)\}^2+(y-2)^2}=5.$$

This relation shows that the distance of the point (x, y) from the fixed point $(-3, 2)$ is 5. [See § *1·2*]. Thus the different positions of the point (x, y) must be at a fixed distance 5 from the fixed point $(-3, 2)$ This identifies the locus of the point (x, y) to be a circle with centre $(-3, 2)$ and radius 5. This is therefore the locus represented by the equation.

Conversely, suppose a straight line on a plane with chosen axes be given, passing through the given points (3, 2) and (− 1, 5). Then if (x, y) be the co-ordinates of any arbitrary point on the line, the area of the triangle with (x, y), (3, 2) and (− 1, 5) as vertices must be zero.

Hence, $\frac{1}{2}\{x(2-5)+3(5-y)+(-1)(y-2)\}=0.$ [See § *1·4*]

i,e., $-3x-4y+17=0,$ or, $3x+4y=17.$

This being the relation satisfied by the co-ordinates of any and every point on the given line, it represents the equation to the given line.

2·2. Every first degree equation in x, y must represent a straight line.

Let $ax+by+c=0$ ··· (i) be a given equation of the first degree. Clearly, any equation of the first degree in x, y is of this type.

Take any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the locus. Any other point $P(x_3, y_3)$ on the locus being taken, the co-ordinates of A, B and P satisfy the given equation.

Hence, $ax_1+by_1+c=0$ ··· (ii)

$ax_2+by_2+c=0$ ··· (iii)

and $ax_3+by_3+c=0$ ··· (iv)

From (ii) and (iii), by cross-multiplication,

$$\frac{a}{y_1-y_2}=\frac{b}{x_2-x_1}=\frac{c}{x_1y_2-x_2y_1}.$$

∴ substituting in (iv),

$$x_3(y_1-y_2)+y_3(x_2-x_1)+(x_1y_2-x_2y_1)=0,$$

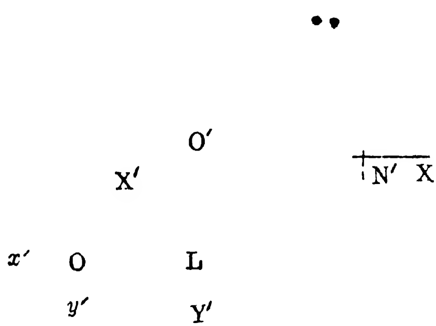
or, $$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

This shows that the triangle ABP is of zero area [See § *1·4*]. Thus, P lies on the straight line AB. As P is any point on the locus represented by (i), it follows that the locus is a straight line.

2·3. Transfer of origin, directions of axes remaining unchanged.

Let O be the origin, and let, xOx', yOy' be a set of rectangular axes, with respect to which co-ordinates of a point P are x, y.

Let the origin be transferred to the point O' whose co-ordinates referred to the given axes xOx' and yOy' are (α, β), and let the axes $XO'X'$, and $YO'Y'$, which are parallel respectively to xOx' and yOy' through O' be taken as the new axes of co-ordinates. Suppose the co-ordinates of P referred to this new set of axes be X, Y.



Let PN be perpendicular to Ox and let it intersect $XO'X'$ at N'. Also let $YO'Y'$ intersect Ox at L.

Then evidently $ON = x$; $NP = y$, $O'N' = X$, $N'P = Y$, $OL = \alpha$, $LO' = \beta$, in magnitude and sign.

Now, from the figure,

$$ON = OL + LN = OL + O'N'$$

and $\quad NP = NN' + N'P = LO' + N'P.$

Hence,
$$\mathbf{x = X + \alpha}$$
$$\mathbf{y = Y + \beta.}$$

These are the equations of transformation, connecting the old co-ordinates of P with the new ones.

These equations are easily seen to be true, in whichever quadrant O' may lie, provided in writing the relations from the figure, we take due notice of the signs of the quantities involved.

Thus, if an equation to a locus or curve be given referred to a set of rectangular axes, and if the origin be transferred to a position whose co-ordinates are α, β, the axes remaining unchanged in direction, the equation of the curve referred to the

new axes will be obtained by replacing in the given equation x by $X+\alpha$ and y by $Y+\beta$.

2·4. Illustrative Examples.

Ex. 1. *Prove that the locus of a point which is always equidistant from two given points is a straight line.*

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two given points A and B, and let (x, y) be the co-ordinates of any point P which is equidistant from them. Then, $PA=PB$ or $PA^2=PB^2$.

$$\therefore \quad (x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2,$$

$$\text{or,} \quad 2(x_2-x_1).x+2(y_2-y_1).y+(x_1^2+y_1^2-x_2^2-y_2^2)=0.$$

This being the equation satisfied by the co-ordinates (x, y) of any position of P, it represents the equation to the locus of P.

As this is a first degree equation in (x, y), it represents a straight line [See § 2·2]

Ex. 2. *A and B are two given points whose co-ordinates are* $(-5, 3)$ *and* $(2, 4)$ *respectively. A point P moves in such a manner that* $PA : PB=3:2$. *Find the equation to the locus traced out by P. What curve does it represent?*

Let (x, y) be the co-ordinates for any position of P.

Then, $PA=\sqrt{(x+5)^2+(y-3)^2}$ and $PB=\sqrt{(x-2)^2+(y-4)^2}$. By the given condition, $PA : PB=3:2$ or $4PA^2=9PB^2$.

$$\therefore \quad 4\{(x+5)^2+(y-3)^2\}=9\{(x-2)^2+(y-4)^2\},$$

whence, $5(x^2+y^2)-76x-48y+44=0$.

As this is the equation satisfied by the co-ordinates of any position of P, it represents the required equation to the locus of P.

The equation may be put in the form

$$x^2+y^2-\tfrac{76}{5}x-\tfrac{48}{5}y+\tfrac{44}{5}=0,$$

$$\text{or,} \quad (x-\tfrac{38}{5})^2+(y-\tfrac{24}{5})^2=(\tfrac{38}{5})^2+(\tfrac{24}{5})^2-\tfrac{44}{5}=72,$$

$$\text{or,} \quad \sqrt{(x-7\tfrac{3}{5})^2+(y-4\tfrac{4}{5})^2}=6\sqrt{2}.$$

This shows that the distance of the moving point x, y from the fixed point $(7\frac{3}{5}, 4\frac{4}{5})$ is constant $=6\sqrt{2}$. This identifies the locus to be a circle with centre $(7\frac{3}{5}, 4\frac{4}{5})$ and radius $6\sqrt{2}$.

Ex. 3. *Prove that the equation*

$$9x^2+25y^2-108x+100y+199=0$$

by properly transferring the origin without turning the axes can be reduced to the form

$$\frac{x^2}{25}+\frac{y^2}{9}=1.$$

The given equation can be written as

$$9(x^2-12x)+25(y^2+4y)=-199,$$

or, $9(x-6)^2+25(y+2)^2=9.36+25.4-199=225.$

Now transferring the origin to $(6,-2)$ without turning the axes *i.e.*, replacing x by $x+6$ and y by $y-2$, [See § 2 3] the new equation becomes

$$9x^2+25y^2=225, \text{ or, } \frac{x^2}{25}+\frac{y^2}{9}=1.$$

Examples II

1. A point moves so that (i) its distance from the y-axis is always -3, (ii) the sum of its distances from the axes of co-ordinates is always 9. Write down the equation to its locus in each case.

2. Find the equation to the locus of a point which moves so that

(i) its distance from the point $(-3, 7)$ is always equal to 4.

(ii) its distance from the y-axis is always half its distance from the orign.

(iii) the rectangle contained by its distances from the axes is equal to a square of side c.

3. If the difference of the squares of the distances of a moving point from two fixed points is constant, show that it traces a straight line.

4. A and B are two fixed points whose co-ordinates are $(-3, 4)$ and $(5, -2)$ respectively. A point P moves in such a manner that the area of the triangle PAB remains constant. Show analytically that its locus is a straight line.

5. If A, B, C are three fixed points whose co-ordinates are $(a, 0)$, $(-a, 0)$, $(c, 0)$ respectively, and if P be a movable point such that $PA^2+PB^2=2PC^2$, find the equation to the locus of P.

6. The co-ordinates of three fixed points A, B, C are $(3, 4)$, $(-2, 5)$ and $(-1, -9)$ respectively. P is a variable point such that $PA^2 + PB^2 + PC^2 = \text{constant}$. Prove that P traces out a circle.

7. A and B are two fixed points and P is a movable point such that $PA : PB$ is constant. Show that the locus of P is a circle.

[*Choose the middle point of AB as origin, and AB as x-axis.*]

8. A point moves so that its distance from the fixed point $(2a, 0)$ is equal to its distance from the y-axis. Determine the equation to its locus.

If the origin be transferred to $(a, 0)$, the axes remaining fixed in direction, prove that the transformed equation is $y^2 = 4ax$.

9. The sum of the distances of a movable point P from the two fixed points $(c, 0)$ and $(-c, 0)$ is a constant $= 2a$. Prove that the equation to the locus of P is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 - b^2 = c^2.$$

[Let $P(x, y)$. $\therefore \quad \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$,

or, $\sqrt{(x+c)^2-y^2} = 2a - \sqrt{(x-1)^2+y^2}$,

or, $a^2 - cx = a\sqrt{(x-c)^2+y^2}$.

Squaring, $x^2(a^2-c^2)+a^2y^2 = a^2(a^2-c^2)$,

or, $x^2b^2 + a^2y^2 = a^2b^2$,

or, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is the locus of P.]

10. The distance between two fixed points S and S' is $2c$. Taking the middle point of SS' as origin, and SS' as x-axis, determine the equation to the locus of a point P which moves so that

$$PS - PS' = 2a \ (a \text{ constant}).$$

[If the mid-point of SS' is taken as origin, the co-ordinates of $S(c, O)$ and S' $(-c, O)$ and let $P(x, y)$.

$PS - PS' = 2a$ *i.e.*, $\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = 20$,

or, $\sqrt{(x-c)^2+y^2} = 2a + \sqrt{(x+c)^2+y^2}$.

Proceed as in Ex. 9.]

11. Transform to parallel axes through the point $(1, -2)$ the equations

(i) $y^2 - 4x + 4y + 8 = 0$. (iii) $2x^2 + y^2 - 4x + 4y = 0$.

12. What does the equation

$$(a - b)(x^2 + y^2) - 2abx = 0$$

become if the origin be transferrd to $\dfrac{ab}{a-b}, 0$, axes remaining unchanged in direction ?

13. The equation to a curve being

$$(1 - e^2)\, x^2 + y^2 + d^2 = 2dx,$$

find its equation if the origin be shifted to $\dfrac{d}{1-e^2}, 0$, without rotation of axes.

[Replacing x, y by $x + \dfrac{d}{1-e^2}$, y equation becomes $(1-e^2)\left[\left(x + \dfrac{d}{1-e^2}\right)^2\right] + y$
$+ d^2 = 2d\left(x + \dfrac{d}{1-e^2}\right)$,

or, $(1-e^2)\, x^2 + y^2 = \dfrac{d^2 e^2}{1-e^2}$.]

14. By transforming to parallel axes through a properly chosen point, prove that the equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

can be reduced to a homogeneous equation of the second degree in x, y.

ANSWERS

1. (i) $x + 3 = 0$. (ii) $x + y = 9$. **2.** (i) $x^2 + y^2 + 6x - 14y + 42 = 0$.

(ii) $y = \pm x\sqrt{3}$. (iii) $xy = c^2$. **5.** $x = \dfrac{c^2 - a^2}{2c}$.

8. $y^2 = 4a(x - a)$. **10.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$. **11.** (i) $y^2 = 4x$.

(ii) $\dfrac{x^2}{3} + \dfrac{y^2}{6} = 1$. **12.** $x^2 + y^2 = \left(\dfrac{ab}{a-b}\right)^2$. **13.** $x^2 + \dfrac{y^2}{1-e^2} = \left(\dfrac{de}{1-e^2}\right)^2$.

CHAPTER III

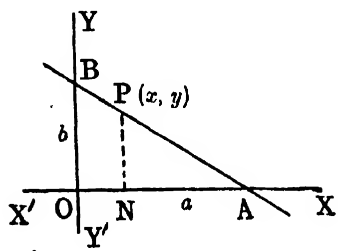
STRAIGHT LINE

3·1. Standard forms of the equation.

In the previous chapter we have seen that a first degree equation always represents a straight line. Now the position of a given straight line on a plane may be fixed up in various ways, and according to each way of defining the position of a line, we get a definite form of the equation to the line. The standard forms of the equation to a line are given below :

(A) *Intercepts on the axes are given in magnitude and sign.*

Let a straight line AB intersect the axes of x and y at A and B respectively, and let the *intercepts* OA and OB on the axes be a and b (given in magnitude and sign) so that the position of the line is definite.



Let (x, y) denote the co-ordinates of any point P on the line. Let PN be the perpendicular from P on OX, so that $ON = x$, $NP = y$.

Now the triangles PNA and BOA are easily seen to be similar.

$$\therefore \frac{NP}{OB} = \frac{NA}{OA}, \text{ or } \frac{y}{b} = \frac{a-x}{a} = 1 - \frac{x}{a}.$$

$$\therefore \frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = \mathbf{1}$$

which being the relation satisfied by the co-ordinates (x, y) of any point on the line, is the equation to the line.

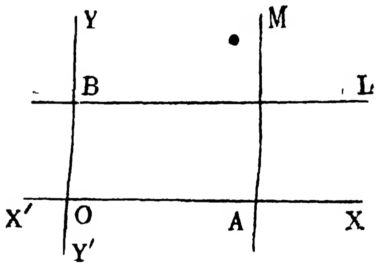
Alternatively

The co-ordinates of A and B are evidently (a, o) and (o, b) respectively. Now P being on the straight line AB, the triangle PAB is of zero area.

Hence, $\frac{1}{2}\{x(0-b)+a(b-y)+0(y-0)\}=0$

or, $-xb+ab-ay=0$, whence $\frac{x}{a}+\frac{y}{b}=1$.

(B) *Straight lines parallel to x-axis or y-axis.*



For a straight line BL parallel to the x-axis, if its intercept $OB=b$, then clearly the distance of every point of it from the x-axis being the same, namely b, the y-co-ordinate of every point of it is b, whatever its x-co-ordinate may be. Hence, its equation is

$$\mathbf{y=b.}$$

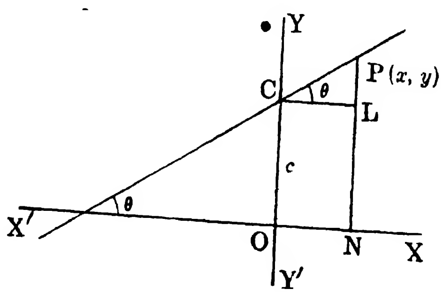
Similarly, for a straight line AM parallel to the y-axis, if its intercept OA on the x-axis be a, the equation will be

$$\mathbf{x=a.}$$

Conversely, equations of the form $x=a$, or $y=b$, in which one co-ordinate is absent, represent straight lines parallel to the y-axis or to the x-axis, as the case may be.

Note. It may be noted that when a straight line becomes parallel to the y-axis, its intercept on the y-axis, $b \to \infty$, and so the equation to the line, from (A), reduces to the form $\frac{x}{a}=1$, *i.e.*, $x=a$. Similarly, for lines parallel to the x-axis, the equation reduces to the form $y=b$.

(C) *Intercept on the y-axis, and inclination to the x-axis are given.*



Let a straight line, inclined at a given angle θ to the x-*axis*, intersect to the y-axis at C, and let the intercept OC be given to

be c (in magnitude and sign). θ and c being given, the position of the line is definitely fixed.

Let (x, y) be the co-ordinates of any point P on the line. PN being perpendicular from P on OX, we have $ON = x$, $NP = y$. Now CL being perpendicular from C on NP, $LP = NP - NL = NP - OC = y - c$, and $CL = ON = x$. Also $\angle PCL = \theta$ $(\neq \frac{1}{2}\pi)$.

$$\tan\theta = \frac{LP}{CL} = \frac{y-c}{x}. \qquad y = x\tan\theta + c.$$

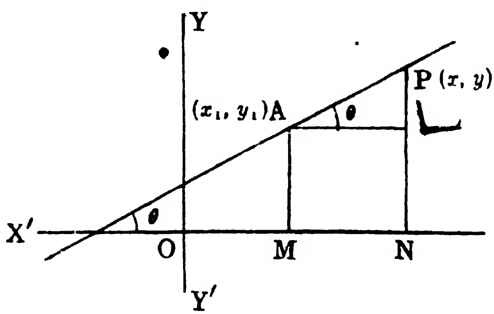
Denoting $\tan\theta$ by m,

$$\mathbf{y = mx + c\ (m = \tan\theta)\ (\theta \neq \tfrac{1}{2}\pi).}$$

This being the relation satisfied by the co-ordinates of any point on the line, it represents the equation to the line.

Note. $m(=\tan\theta)$ is defined as the *slope* or *gradient* of the line. The angle θ may be acute or obtuse, and hence m may be positive or negative according to the position of the line. When the line is parallel to the x-axis, $\theta = 0$, and so $m = 0$, the equation to the line then reduces to the form $y = c$ [See *(B) above*]. If $\theta = 90°$, $m \to \infty$ or $\frac{1}{m} \to 0$, the equation to the line reduces to the form $x = a$ in this case [See *(B) above*].

(D) *Straight line through a given point* (x_1, y_1), *and making a given angle* θ *with the x-axis.*



Let A be the given point (x_1, y_1) and P any point (x, y) on the line, whose inclination to the x-axis is θ.

Let AM, PN be perpendiculars on OX, and AL perpendicular on PN. Then, $\angle PAL = \theta$.

Now, $\tan\theta = \dfrac{LP}{AL} = \dfrac{LP}{MN} = \dfrac{NP - MA}{ON - OM} = \dfrac{y - y_1}{x - x_1}$.

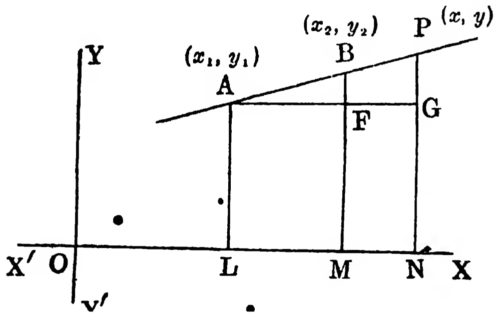
$\therefore \quad \mathbf{y - y_1} = (x - x_1)\tan\theta = \mathbf{m(x - x_1)}$, where $m = \tan\theta$,

or, $\quad \dfrac{\mathbf{x - x_1}}{\mathbf{\cos\theta}} = \dfrac{\mathbf{y - y_1}}{\mathbf{\sin\theta}}$,

any one of which represents the required equation to the line.

(E) *Straight line passing through two given points* (x_1, y_1) *and* (x_2, y_2).

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points through which a straight line passes, and $P(x, y)$ be any point on it.



Draw AL, BM, PN perpendiculars on OX, and let AFG be drawn parallel to OX, intersecting BM and PN at F and G respectively.

Then, triangles PAG and BAF are evidently similar.

$$\frac{AG}{AF} = \frac{GP}{FB}, \quad \text{or,} \quad \frac{ON - OL}{OM - OL} = \frac{NP - LA}{MB - LA}.$$

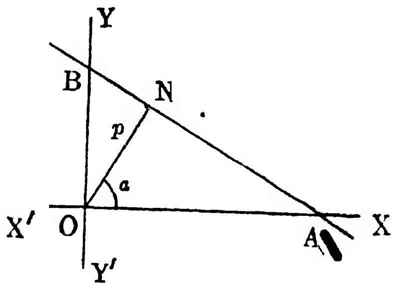
$$\therefore \quad \frac{\mathbf{x - x_1}}{\mathbf{x_2 - x_1}} = \frac{\mathbf{y - y_1}}{\mathbf{y_2 - y_1}},$$

$$\text{or,} \quad \mathbf{y - y_1} = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}\,\mathbf{(x - x_1)},$$

any one of which represents the required equation to the line.

The slope of the line, $m=\dfrac{y_2-y_1}{x_2-x_1}$ in this case.

(F) *Perpendicular distance p from the origin on the line, and the angle α which this perpendicular makes with the x-axis are*



Let the straight line intersect the axes at A and B, and let p be the perpendicular ON to it from the origin, and α the angle XON made by ON with the x-axis.

Clearly,

$OA=ON \sec AON=p \sec \alpha$,

and $\quad OB=ON \sec NOB=p \sec (90^\circ-\alpha)=p \operatorname{cosec} \alpha$.

These being the intercepts on the axes, the equation to the line is

$$\frac{x}{p \sec \alpha}+\frac{y}{p \operatorname{cosec} \alpha}=1 \text{ or } \mathbf{x \cos \alpha+y \sin \alpha=p.}$$

Note. Here p is always taken with a positive sign. The angle α however may have any value, positive or negative, acute or obtuse. According to the quadrant in which ON lies, the signs of $\cos \alpha$ and $\sin \alpha$ will become definite.

3·2. Reduction of the most general equation ax+by+c=0 of the first degree of a straight line to standard forms.

It may be noted that the most general form of the equation of a straight line, namely $ax+by+c=0$ can be reduced to any of the standard forms given above.

For instance, the above equation may be written as $ax+by=-c$,

or, $\quad \dfrac{y}{}$ $\dfrac{c}{a}$

which is of the form (A), showing that the intercepts on the axes of x and y are $-\dfrac{c}{a}$ and $-\dfrac{c}{b}$ respectively.

Again, we can write the above equation as $by = -ax - c$,

or, $y = -\dfrac{a}{b}x - \dfrac{c}{b}$

which is of the form (C), showing that its slope on the x-axis, or, $m = -\dfrac{a}{b}$, and its intercept on the y-axis is $-\dfrac{c}{b}$.

Further, the general equation $ax + by + c = 0$ can be written as $ax + by = -c$, or dividing by $\pm\sqrt{a^2+b^2}$, we can write it as

$$\frac{a}{\pm\sqrt{a^2+b^2}}x + \frac{b}{\pm\sqrt{a^2+b^2}}y = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

which is of the form (F), as can be evidenced by writing $\cos\alpha = \dfrac{a}{\pm\sqrt{a^2+b^2}}$, whence $\sin\alpha = \dfrac{b}{\pm\sqrt{a^2+b^2}}$. The perpendicular from the origin on the line, $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$ (where *that sign in the denominator is to be taken which makes the perpendicular p positive in sign*). The sign of the denominator being thus fixed, the angle α is given by $\cos\alpha = \dfrac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\pm\sqrt{a^2+b^2}}$, which will have their magnitudes and signs definite, and accordingly α will be definitely known, *i.e.*, the quadrant in which the perpendicular from the origin on the line falls is known and p being known, the position of the line is fixed up.

Note. That the intercepts of the line $ax+by+c=0$ on the axes are $-c/a$ and $-c/b$ can also be obtained by putting $y=0$ and $x=0$ respectively in the equation.

3·3. Point of intersection of two given lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

At the intersection point, both the equations being satisfied, we get by solving the equations as simultaneous equations,

$\overline{\mathbf{b_1c_2-b_2c_1}} = \overline{\mathbf{c_1a_2-c_2a_1}} = \overline{\mathbf{a_1b_2-a_2b_1}}$ ∴ giving the co-ordinates of the point of intersection.

$$\mathbf{x}=\frac{\mathbf{b_1c_2-b_2c_1}}{\mathbf{a_1b_2-a_2b_1}},\ \mathbf{y}=\frac{\mathbf{c_1a_2-c_2a_1}}{\mathbf{a_1b_2-a_2b_1}}\quad\left[\frac{a}{a_2}\neq\frac{c_1}{b_2}\right]$$

3·4. Equation of a straight line through the point of intersection of two given lines $\mathbf{a_1x+b_1y+c_1=0}$, $\mathbf{a_2x+b_2y+c_2=0}$.

(i) The co-ordinates of the point of intersection of the two given lines, by solving the simultaneous equations (as in § *3·3*) are

$$x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ y_1=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Now any line through x_1, y_1 is $y-y_1=m(x-x_1)$ where m is arbitrary and different for different lines.

(ii) The equation $\mathbf{(a_1x+b_1y+c_1)+k(a_2x+b_2y+c_2)=0}$ $k\neq 0$ is clearly a first degree equation, and thus represents a straight line. Again, at the intersection point of the two given lines the above equation is satisfied for all values of k, since each of the bracketed portions is zero there. Thus the equation always represents a straight line passing through the intersection of the two given lines. For different values of k, it represents different such lines.

3·5. Condition that three given lines, $\mathbf{a_1x+b_1y+c_1=0}$, $\mathbf{a_2x+b_2y+c_2=0}$, $\mathbf{a_3x+b_3y+c_3=0}$ may be concurrent.

The point of intersection of the last two lines (by solving them as simultaneous equations) is easily seen to have co-ordinates

$$x=\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2},\ y=\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}.$$

The first line will pass through this point only if the above co-ordinates satisfy its equation. Hence, the required condition that the three lines may be concurrent is

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)+c_1=0$$

or, $\mathbf{a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0}$. (A

Alternative condition :

If any three non-zero constants l, m, n can be found such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0. \quad \text{(B)}$$

Identically, then the above three lines are concurrent.

If (x', y') be the point of intersection of the first two lines, then $a_1x'+b_1y'+c_1=0$ and $a_2x'+b_2y'+c_2=0$ and hence substituting x', y' for x, y, in the given relation (B), we get

$$n(a_3x'+b_3y'+c_3)=0, \text{ i.e., } a_3x'+b_3y'+c_3=0$$

i.e., the third line $a_3x+b_3y+c_3=0$, passes through (x', y'), the point of intersection of the first and second *i.e.*, the three lines are concurrent.

3·6. Condition that three given points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear.

A, *B*, *C* denoting the given points, if they be collinear, the triangle formed by them is of zero area. For this the condition is [from § *1·4*]

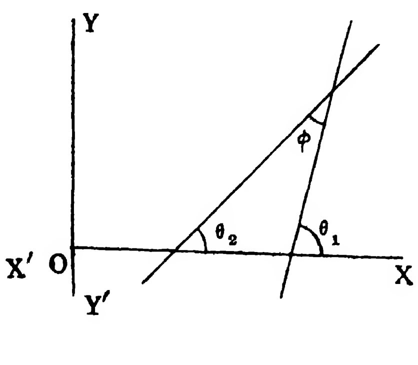
$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

Note. The above condition can also be deduced by writing down the equation of the straight line through any two of the points, and using the condition that this equation is satisfied by the co-ordinates of the third point.

3·7. Angle between two given lines.

(A) *When the lines are given by the equations*

$$y=m_1x+c_1, \quad y=m_2x+c_2.$$



Let θ_1 and θ_2 be the angles made by the given lines with the x-axis.

Then,

$$\tan\theta_1=m_1, \quad \tan\theta_2=m_2$$

Now ϕ being the angle of intersection between the two lines, clearly

$$\phi=\theta_1\sim\theta_2.$$

$$\therefore \quad \tan\phi = \frac{\tan\theta_1 \sim \tan\theta_2}{1+\tan\theta_1 \tan\theta_2} = \frac{m_1 \sim m_2}{1+m_1 m_2}.$$

Thus, ϕ is known.

(B) *When the lines are given by the equations*

$$a_1x + b_1y + c_1 = 0 \textit{ and } a_2x + b_2y + c_2 = 0.$$

Here the slopes of the lines are clearly given by

$$m_1 = -\frac{a_1}{b_1},\ m_2 = -\frac{a_2}{b_2}.$$

Hence, by (A) above,

$$\tan\phi = \frac{\left(-\frac{a_1}{b_1}\right) \sim \left(-\frac{a_2}{b_2}\right)}{1+\frac{a_1a_2}{b_1b_2}} = \frac{a_1b_2 \sim a_2b_1}{a_1a_2+b_1b_2}.$$

Cor. The two lines will be *parallel* to one another when $\phi=0$; hence the condition of parallelism is

$$m_1 = m_2 \quad \text{or,} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

The two lines will be *perpendicular* to one another when $\phi = \frac{1}{2}\pi$ or $\cot\phi = 0$, giving

$$1+m_1m_2=0, \quad \text{or,} \quad m_1m_2=-1, \quad \text{or,} \quad a_1a_2+b_1b_2=0$$

as the necessary condition.

Note. From above, it is clear that any straight line parallel to $y=m_1x+c_1$ may have its equation taken as

$$y=m_1x+c_2, \text{ where } c_2 \text{ is arbitrary.}$$

Similarly, any line parallel to $a_1x+b_1y+c_1=0$ may have its equation assumed in the form $a_1x+b_1y+c_2=0$, where c_2 is different for different such lines.

Again, any line perpendicular to $y=m_1x+c_1$ can be taken to be $y=-\frac{1}{m_1}x+c_2$, and any line perpendicular to $a_1x+b_1y+c_1=0$ may be assumed as $b_1x-a_1y+c_2=0$, where c_2 is arbitrary in both cases.

These assumptions are very useful in working out examples.

3·8. Illustrative Examples.

Ex. 1. *Find the equation of the straight line passing through the point* $(-5, 3)$ *and perpendicular to the line* $x-2y+6=0$.

Any line through $(-5, 3)$ has its equation of the form [See § *3·1, D*]

$$y-3=m(x+5) \quad \cdots \quad \cdots \quad \text{(i)}$$

The given line is $2y=x+6$ or $y=\frac{1}{2}x+3$, showing that its slope 'm' is $\frac{1}{2}$.

Now, (i) being perpendicular to this line, the necessary condition gives

$$m \times \tfrac{1}{2} = -1 \text{ or } m=-2.$$

Hence, (i) becomes

$$y-3=-2(x+5), \text{ or, } 2x+y+7=0$$

which is the equation of the required line.

Ex. 2. *Obtain the equation of the straight line which passes through the intersection of the lines* $2x-3y+4=0$ *and* $3x+4y-5=0$, *and has equal intercepts of the same sign along the axes. Find the perpendicular distance from the origin on this straight line.*

Any line passing through the intersection of the two given lines can have its equation written in the form

$$2x-3y+4+k(3x+4y-5)=0,$$

$$\text{or, } (2+3k)x+(4k-3)y+(4-5k)=0 \quad \cdots \quad \text{(i)}$$

Its intercept on the x-axis (by putting $y=0$ in the above equation) is $\dfrac{5k-4}{2+3k}$.

Similarly, (putting $x=0$) the intercept on the y-axis is $\dfrac{5k-4}{4k-3}$.

As the two intercepts are equal in magnitude, and are of the same sign, we have

$$\frac{5k-4}{2+3k}=\frac{5k-4}{4k-3}, \text{ or, } 2+3k=4k-3, \text{ whence } k=5.$$

Substituting in (i), the equation of the required straight line is

$$17x+17y-21=0, \text{ or, } 17x+17y=21 \quad \cdots \quad \text{(ii)}$$

If p denotes the perpendicular distance on it from the origin, and if α be

the angle which this perpendicular makes with the positive direction of the x-axis, then the equation of the line can be written as

$$x \cos a + y \sin a = p \qquad [\text{ See § } 3{\cdot}1\ (F)\ ]$$

As this is identical with equation (ii), comparing coefficients, we get

$$\frac{\cos a}{17} = \frac{\sin a}{17} = \frac{p}{21}. \qquad \cos a = \frac{17p}{21}, \ \sin a = \frac{17p}{21}.$$

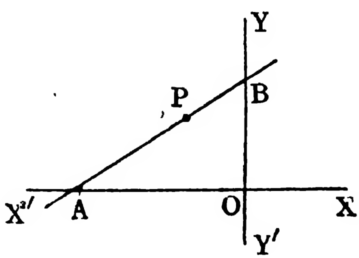
$$\therefore \quad \left(\frac{17p}{21}\right)^2 + \left(\frac{17p}{21}\right)^2 = 1 \text{ giving } p^2 = \frac{21^2}{17^2 + 17^2}$$

whence, $p = \frac{21}{17\sqrt{2}}$ 21 [illegible] $\sqrt{2}$.

As the perpendicular from the origin on a line is always taken as positive, the positive sign of the square root is taken.

Ex. 3. *Find the equation to the straight line which passes through the point* $(-4, 3)$ *and is such that the portion of the line between the axes is divided at the point internally in the ratio* $5 : 3$.

Let P be the point $(-4, 3)$, and let APB be the line intersecting the axes at A and B, such that $AP : PB = 5 : 3$.



Let the intercepts OA and OB on the axes be a and b respectively in magnitude and sign. Then co-ordinates of A are evidently $(a, 0)$ and those of B are $(0, b)$.

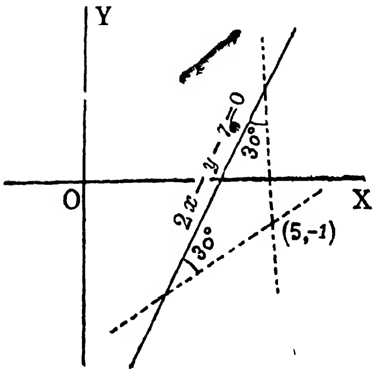
The co-ordinates of the point P which divides AB in the ratio $5 : 3$ are then

$$\frac{5.0 + 3.a}{5+3} \text{ and } \frac{5.b + 3.0}{5+3} \ i.e., \ \frac{3a}{8} \text{ and } \frac{5b}{8}.$$

Thus, $\frac{3a}{8} = -4$ and $\frac{5b}{8} = 3$. Hence, $a = -\frac{32}{3}$, $b = \frac{24}{5}$. The equation to the line AB then, having intercepts $-\frac{32}{3}$ and $\frac{24}{5}$ along the axes is

$$\frac{3x}{-32} + \frac{5y}{24} = 1, \ i.e., \ 9x - 20y + 96 = 0.$$

Ex. 4. *Find the equations to the straight lines passing through the point $(5, -1)$ and making an angle of $30°$ with the line $2x - y - 7 = 0$.*



Any line passing through $(5, -1)$ is $y + 1 = m(x - 5)$ ··· (i). The given line

$$\therefore \quad y = 2x - 7$$

has its 'm' $= 2$.

If (i) makes an angle $30°$ with it, we have

$$\tan 30° = \frac{2 \sim m}{1 + 2m}.$$

Thus, $\dfrac{1}{\sqrt{3}} = \dfrac{2 - m}{1 + 2m}$, or $\dfrac{m - 2}{1 + 2m}$.

$$= \frac{2\sqrt{3} - 1}{2 + \sqrt{3}}, \text{ or } -\frac{2\sqrt{3} + 1}{2 - \sqrt{3}}.$$

Hence, the equation to the required lines are

$$y + 1 = \frac{2\sqrt{3} - 1}{2 + \sqrt{3}}(x - 5), \text{ and } y + 1 = -\frac{2\sqrt{3} + 1}{2 - \sqrt{3}}(x - 5).$$

Examples III(a)

1. Obtain the equation to the line inclined to the x-axis at an angle $60°$, and having its intercept on the x-axis equal to -1.

2. Find the equation to the line passing through the point $(-7, 3)$ and having its intercept on the x-axis equal to -5.

3. Find the equation to a straight line passing through the point $(-2, -3)$, and having equal intercepts of opposite sign on the axes.

4. Find the equations to the straight lines joining the pair of points

(i) $(-3, 7)$ and $(-1, -2)$; (ii) $(0, -3)$ and $(7, 0)$;

(iii) $(-2, 6)$ and $(-2, -4)$.

5. Show that the equations of the straight lines joining the points

(i) $(at_1{}^2, 2at_1)$ and $(at_2{}^2, 2at_2)$ is

$$y(t_1+t_2)-2x=2at_1t_2.$$

(ii) $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta', b\sin\theta')$ is

$$\frac{x}{a}\cos\frac{1}{2}(\theta+\theta')+\frac{y}{b}\sin\frac{1}{2}(\theta+\theta')=\cos\frac{1}{2}(\theta-\theta').$$

6. Find the equation to the straight line joining the origin to the point of intersection of the lines $2x-5y=1$ and $x+3y+7=0$.

7. Find the equation to the straight line passing through the point $(-2, 1)$ and also through the intersection of the lines $2x-3y+5=0$ and $x+4y-7=0$.

8. (i) Obtain the equation to the straight line passing through the point $(-1, 2)$ and perpendicular to the line $3x+4y=5$.

[*H. S. 1960, Compartmental*]

[Equation of line through

$(-1, 2)$ is $y-2=m(x+1)$.

Since this line $\perp$ to

$3x+4y=5$, hence $mx-\frac{3}{4}=-1$. [§ 3·7]

Putting the value of m and simplifying, equation of line is obtained.

$\therefore\ m=\frac{4}{3}$.]

(ii) Find the co-ordinates of the foot of the perpendicular from the point $(8, -6)$ on the straight line $2x-3y+5=0$.

9. (i) Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear.

(ii) Prove that the points whose co-ordinates are respectively $(5, 1)$, $(1, -1)$ and $(11, 4)$ lie on a straight line and find the intercepts of this line on the axes. [*H. S. 1961, Comp.*]

[Equation of this line through

$(5, 1)$ and $(1, -1)$ is $\frac{x-5}{1-5}=\frac{y-1}{-1-1}$ or $x-2y-3=0$

which is satisfied by $(11, 4)$. Hence the points are collinear.]

10. If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear, show that their join passes through the origin, and $ab'=a'b$.

11. (i) Prove that the three lines $2x-7y+10=0$, $3x-2y-2=0$ and $x-7y+12=0$ meet in a point.

(ii) Find for what value of k the three lines $x-y+5=0$, $x+y=1$ and $y=kx+13$ will be concurrent.

Find also the point of concurrence.

✓12. Find the equation to the perpendicular bisector of the straight line joining the points $(-2, 7)$ and $(8, -1)$. At what distance is this perpendicular bisector from the origin? [*H. S. 1961*]

13. Determine the equation of the straight line which passes through the intersection of the lines given by $3x-4y+1=0$ and $5x+y=1$, and has equal intercepts of the same sign along the axes. [*H. S. 1960*]

14. A straight line passes through the intersection of the lines $3x-7y+5=0$, $x-2y-7=0$, and has equal intercepts of the same sign along the axes. Find the length of the perpendicular on it from the origin. [*H. S. 1961, Compartmental*]

15. A straight line is drawn through the point $(3, 5)$ such that the point bisects the portion of the line intercepted between the axes. Find the equation to the line, and calculate its perpendicular distance from the origin. [*H. S. 1960, Comp.*]

16. The portion of a straight line intercepted between the axes is divided internally in the ratio $2:3$ at the point $(-9, -2)$. Show that it passes through the point $(0, -5)$ and is perpendicular to the line $y=3x$.

17. (i) Find the equation of the line joining the point of intersection of the lines $x+3y+2=0$, $2x-y-3=0$ to the point of intersection of $7x-y-3=0$, $2x-5y-15=0$.

(ii) Find the equations of the diagonals of the parallelogram formed by the lines $4x-5y-7=0$, $4x-5y-14=0$, $3x+7y-8=0$, $3x+7y-12=0$.

18. Determine the angle between the lines $y=\frac{1}{2}(x-5)$ and $+3y=2$. Also find out the equation to the line through their point of intersection making a positive angle 60° with the positive direction of the x-axis.

19. Show that the lines $x \cos a + y \sin a = p$,

$$x \cos (a + \tfrac{2}{3}\pi) + y \sin (a + \tfrac{2}{3}\pi) = p \text{ and}$$

$$x \cos (a - \tfrac{2}{3}\pi) + y \sin (a - \tfrac{2}{3}\pi) = p$$ form an equilateral triangle.

[*Angle between each pair is 60°*]

[As in § 3·7(A), from the first two lines

$$m_1 = -\cot a \quad \text{and} \quad m_2 = -\cot \left(a + \frac{2}{3}\pi\right).$$

$$\therefore \quad \tan \theta = \frac{-\cot a + \cot (a + \frac{2}{3}\pi)}{1 + \cot a \cot (a + \frac{2}{3}\pi)} = \frac{1}{\cot [a - (a + \frac{2}{3}\pi)]} = -\tan (\tfrac{2}{3}\pi)$$

$$= -\tan (180° - 60°) = \tan 60°. \qquad \therefore \quad \theta = 60°.$$

Again considering the last two lines, we have

$$\tan \theta = \frac{-\cot (a + \frac{2}{3}\pi) + \cot (a - \frac{2}{3}\pi)}{1 + \cot (a + \frac{2}{3}\pi) \cot (a - \frac{2}{3}\pi)}$$

$$\overline{\cot [(a + \tfrac{2}{3}\pi) - (a - \tfrac{2}{3}\pi)]}$$

$$= \frac{1}{\cot \frac{4}{3}\pi} = \tan (\pi + \tfrac{1}{3}\pi) = \tan \tfrac{1}{3}\pi$$

$$= \tan 60°$$

$$\therefore \quad \theta = 60°$$

Thus each of the two angles of a triangle being 60°, the third angle must be 60°. Hence the triangle is equilateral.

20. Find the equations of the two lines through the point $(3, -1)$ which make an angle $45°$ with the line $2x - y = 2$.

21. In any triangle, show analytically that

(i) the medians are concurrent

[We take BC as x-axis and the mid-point of BC, say D, as origin. Let the co-ordinates of B, C be $(-a, 0)$ and $(a, 0)$. Equation of AD is

$$\frac{x - x_1}{0 - x_1} = \frac{y - y_1}{0 - y_1}, \quad \text{or,} \quad xy_1 - x_1 y = x_1 y_1 - x_1 y_1 = 0. \quad \cdots \quad (1)$$

Mid-point of AC, say E, has co-ordinates $\left(\frac{x_1+a}{2}, \frac{y_1}{2}\right)$.

Equation of BE is $\dfrac{x+a}{\frac{x_1+a}{2}-a}=\dfrac{y-0}{\frac{y_1}{2}-0}$,

or, $\dfrac{x+a}{x_1+3a}=\dfrac{y}{y_1}$, or, $xy_1-y(x_1+3a)+ay_1=0$. ... (2)

Mid-point of AB is $\left(\frac{x_1-a}{2}, \frac{y_1}{2}\right)$.

Equation of CF is $\dfrac{x-a}{\frac{x_1-a}{2}-a}=\dfrac{y-0}{\frac{y_1}{2}-0}$

or, $xy_1-y(x_1-3a)-ay_1=0$.

Solving the last two equations, the co-ordinates of the point of intersection of two medians are obtained and the co-ordinates are $(\frac{1}{3}x_1, \frac{1}{3}y_1)$. But $(\frac{1}{3}x_1, \frac{1}{3}y_1)$ is satisfied by the equation (1). Hence they are concurrent.]

(ii) the perpendicular bisectors of the sides are concurrent.

[Taking the co-ordinates of vertices of $\triangle ABC$ as in Ex. 21 (i), equation of line through D perpendicular to BC is $x=0$. ... (1)

Equation of line AB is $\dfrac{x+a}{x_1+a}=\dfrac{y-0}{y_1-0}$, or, $xy_1-y(x_1+a)+ay_1=0$.

$\left[\text{'}m\text{' of this line}=\dfrac{y_1}{x_1+a}\right]$

Equation of line through F (mid-point of AB) is $y-\dfrac{y_1}{2}=m\left(x-\dfrac{x_1-a}{2}\right)$.

If this line is perpendicular to AB, $m\dfrac{y_1}{x_1+a}=-1$. $\therefore$ $m=-\dfrac{x_1+a}{y_1}$.

Hence equation of line through F perpendicular to AB is

$$y-\frac{y_1}{2}=-\frac{x_1+a}{y_1}\left(x-\frac{x_1-a}{2}\right), \text{ or, } 2yy_1+2x(x_1+a)=y_1^2+x_1^2-a^2 \quad \cdots (2)$$

Similarly equation of line through E (mid-point of AC) and perpendicular to AC is

$$y-\frac{y_1}{2}=-\frac{x_1-a}{y_1}\left(x-\frac{x_1+a}{2}\right) \text{ or } 2yy_1+2x(x_1+a)=x^2+y_1^2-a^2 \quad \cdots (3)$$

Solving (1) and (2), the point of intersection of the two bisectors is $\left(0, \dfrac{x_1^2+y_1^2-a^2}{2y_1}\right)$ which is satisfied by equation (3). Hence perpendicular bisectors are concurrent.]

(iii) the perpendiculars from the vertices on opposite sides are concurrent.

[Taking the co-ordinates of vertices of $\triangle ABC$ as in Ex. 21 (i), equation of line AB is $xy_1 - y(x_1+a) + ay_1 = 0$.

$$\left[\text{as in Ex. 21 (ii), 'm' of this line is } \frac{y_1}{x_1+a}\right]$$

Hence equation of line through C perpendicular to AB is

$$y-0=m(x-a), \text{ or, } y=-\frac{x_1+a}{y_1}(x-a) \left[\text{ since } m\frac{y_1}{x_1+a}=-1\right]$$

or, $yy_1+(x-a)(x_1+a)=0 \quad \cdots \quad (1)$

Similarly equation of line through B perpendicular to AC is

$$yy_1+(x+a)(x_1+a)=0 \quad \cdots \quad (2)$$

Equation of line through A perpendicular to BC is $x=x_1 \quad \cdots \quad (3)$

Solving (1) and (3), the point of intersection is $\left(x_1, -\frac{x_1^2-a^2}{y_1}\right)$ which is satisfied by (2). Hence the result.]

22. Find the ortho-centre of the triangle

(i) whose vertices are $(2, 7)$, $(-6, 1)$ and $(4, -5)$.

(ii) whose sides have equations $2x+7y+24=0$, $4x+y=4$, and $x-3y=1$.

23. Find the area of the triangle formed by the straight lines whose equations are $x+2y-5=0$, $y+2x-7=0$ and $x-y+1=0$.

Find also the co-ordinates of the circum-centre of the triangle.

[Solving the equations pair-wise the co ordinates of three vertices A, B, C of the triangle are $(3, 1)$, $(2, 3)$ and $(1, 2)$.

From § 1·4, area of $\triangle ABC = \frac{1}{2}[3(3-2)+2(2-1)+1(1-3)] = \frac{3}{2}$.

Let $P(x, y)$ be the co-ordinates of circum-centre. Since circum-centre is equidistant from three vertices of $\triangle ABC$, hence

$$(x-3)^2+(y-1)^2=(x-2)^2+(y-3)^2=(x-1)^2+(y-2)^2$$

which gives on simplification $2x-4y+3=0$, $x+y-4=0$. Solving, the co-ordinates of the circum-centre are $(\frac{13}{6}, \frac{11}{6})$.]

24. Show that the feet of the perpendiculars from the origin to the lines

$$x+y-4=0,\ x+5y-26=0,\ 15x-27y-424=0$$

all lie on a straight line, and find the equation of this line.

['m' of the line $x+y-4=0$ is -1.

Equation of line through (0, 0) perpendicular to $x+y-4=0$ is

$y=mx=x$ [since $mx-1=-1$. $\therefore m=1$] or, $y-x=0$.

Similarly equation of line through (0, 0) perpendicular to $x+5y-6=0$ i $y-5x=0$.........(0, 0) perpendicular to $15x-27y-424=0$ is $27x+15y=0$.

Solving $y-x=0$ and $x+y-4=0$, the foot of the perpendicular is (2, 2).

Solving $y-5x=0$ and $x+5y-6=0$, the foot of the perpendicular is (1, 5).

Solving $27x+15y=0$ and $15x+27y-424=0$, the foot of the perpendicular is $(\frac{20}{3}, -12)$.

Equation of line through (2, 2) and (1, 5) is $\frac{x-2}{1-2}=\frac{y-2}{5-2}$, or, $3x+y=8$ which is satisfied by $(\frac{20}{3}, -12)$ and hence lie on one straight line.]

25. A line moves so that the sum of the reciprocals of its intercepts on the two axes is constant. Show that it always passes through a fixed point.

[Let the equation of line be $\frac{x}{a}+\frac{y}{b}=1$ (as in § 3·1 A)

and $\frac{1}{a}+\frac{1}{b}=\text{const.}=k$, say.

We can write $\frac{x}{a}+\frac{y}{b}=1=\frac{1}{k}; k=\frac{1}{k}\left(\frac{1}{a}+\frac{1}{b}\right)$

or, $\frac{1}{a}\left(x-\frac{1}{k}\right)+\frac{1}{b}\left(y-\frac{1}{k}\right)=0$. Therefore the line always passes through $(1/k, 1/k)$.]

ANSWERS

1. $y=\sqrt{3}(x+1)$. 2. $3x+2y+15=0$. 3. $x-y=1$

4. (i) $9x+2y+13=0$. (ii) $3x-7y=21$. (iii) $x=-2$.

6. $32y=15x$. 7. $8x-23y+39=0$. 8. (i) $4x-3y+10=0$. (ii) 2, 3.

9. (ii) 3 and $-1\frac{1}{2}$. **11.** (ii) $k=5$; $-2, 3$. **12.** $5x-4y=3$; $\frac{}{\sqrt{41}}$.

13. $23x+23y=11$. **14.** $85/\sqrt{2}$. **15.** $\frac{x}{6}+\frac{y}{10}=1$; $\frac{15}{17}\sqrt{34}$.

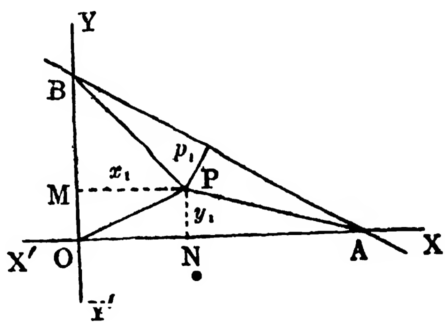
17. (i) $2x-y=3$. (ii) $5x+69y=28$, $37x+29y=112$.

18. $45°$; $5y+3=\sqrt{3}(5x-19)$. **20.** $3x+y=8$, $x-3y=6$.

22. $(-\frac{10}{9}, \frac{49}{27})$; $(\frac{11}{13}, -\frac{7}{13})$. **23.** $\frac{3}{2}$; $(\frac{13}{8}, \frac{11}{6})$. **24.** $3x+y=8$.

3·9. Length of perpendicular from a given point to a given straight line.

(i) *Perpendicular from the point $P(x_1, y_1)$ to the straight line AB given by the equation $ax+by+c=0$.*



A and B being the points of intersection of the line with the axes, from its equation $ax+by+c=0$, the intercepts OA and OB (putting $y=0$ and $x=0$ respectively) are $-\frac{c}{a}$ and

$$\therefore \quad AB=\sqrt{\left(-\frac{c}{a}\right)^2+\left(-\frac{c}{b}\right)^2} \quad \pm\frac{c}{ab}\sqrt{a^2+b^2}.$$

PN and PM being the perpendiculars from $P(x_1, y_1)$ on the axes, $PM=x_1$, $PN=y_1$.

Let p_1 be the required perpendicular from P on AB.

First method :

$$\triangle PAB=\triangle OAB-\triangle POA-\triangle POB.$$

$$\therefore \quad \tfrac{1}{2}p_1.AB=\tfrac{1}{2}OA.OB-\tfrac{1}{2}PN.OA-\tfrac{1}{2}PM.OB,$$

$$\text{or,}\quad p_1.\left(\pm\frac{c}{ab}\sqrt{a^2+b^2}\right)=\left(-\frac{c}{a}\right)\left(-\frac{c}{b}\right)-y_1\left(-\frac{c}{a}\right)-x_1\left(-\frac{c}{b}\right),$$

$$\text{or,}\quad \pm p_1\sqrt{a^2+b^2}=ax_1+by_1+c.$$

$$\therefore\quad \mathbf{p_1=\frac{ax_1+by_1+c}{\pm\sqrt{a^2+b^2}}}.$$

Second method :

P, A, B have co-ordinates (x_1, y_1), $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

$$\therefore\quad \triangle PAB=\frac{1}{2}\left\{x_1\left(0+\frac{c}{b}\right)+\left(-\frac{c}{a}\right)\left(-\frac{c}{b}-y_1\right)+0(y_1-0)\right\}.$$

$$\therefore\quad \tfrac{1}{2}p_1.AB=\frac{1}{2}\left(\frac{cx_1}{b}+\frac{c^2}{ab}+\frac{cy_1}{a}\right),$$

$$\text{or,}\quad p_1\left(\pm\frac{c}{ab}\sqrt{a^2+b^2}\right)=\frac{c(ax_1+by_1+c)}{ab}.$$

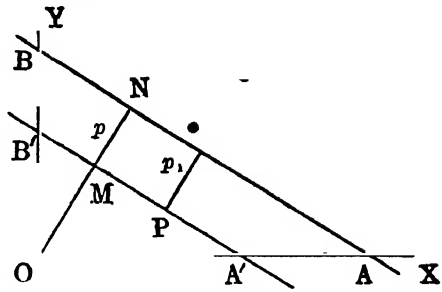
$$\therefore\quad p_1=\frac{ax_1+by_1+c}{\pm\sqrt{a^2+b^2}}.$$

Note. **To choose the proper sign of the denominator, it should be noted that the** ***perpendicular from the origin*** **(0, 0) to the line** ***is always taken as positive.***

∴ the ***sign of the denominator should be taken same as that of c,*** **as is apparent by putting** $x_1=0$, $y_1=0$.

The perpendicular distance of the point P **from the line** AB **is thus positive, if** P **lies on the same side as the origin with respect to the line** AB. **For this case** ax_1+by_1+c **has the same sign as that of** c. **If** P **be on the opposite side of the origin with respect to the line** AB, **the perpendicular from** P **on** AB **will be negative, and in this case,** ax_1+by_1+c **will have its sign opposite to that of** c. **In this connection, see § 3·10.**

(ii) *Perpendicular from the point* $P(x_1, y_1)$ *to the line* $x \cos a + y \sin a = p$.



Here, from the equation to the line, the perpendicular from the origin on it is $ON = p$, and ON makes an angle a with the x-axis.

Let p_1 be the perpendicular from the given point $P(x_1, y_1)$ to the given line AB. If $A'B'$ denotes the parallel line through P, clearly ON is perpendicular to $A'B'$ as well, and the perpendicular OM from O on $A'B'$ is clearly $p - p_1$.

Hence, equation to the line $A'B'$ is

$$x \cos a + y \sin a = p - p_1.$$

As it passes through $P(x_1, y_1)$, we must have

$$x_1 \cos a + y_1 \sin a = p - p_1.$$

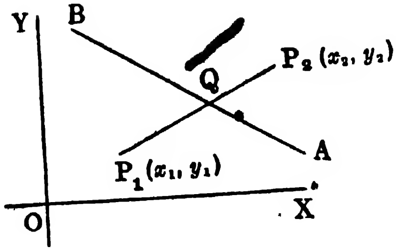
$$\therefore \quad \mathbf{p_1 = p - x_1 \cos \alpha - y_1 \sin \alpha.}$$

Note. Case (i) can be deduced as a corollary to (ii), for the equation to the line being of the form $ax + by + c = 0$, it is reduced to the form of (ii) by dividing by $\pm\sqrt{a^2 + b^2}$ [See § 3·8]. Thus $\dfrac{ax + by + c}{\pm\sqrt{a^2 + b^2}}$ (when proper sign is selected) is of the form $p - x \cos a - y \sin a$, p being positive and $\dfrac{c}{\pm\sqrt{a^2 + b^2}}$ also positive by proper choice of sign.

3·10. Theorem.

The points (x_1, y_1) *and* (x_2, y_2) *are on the same side or opposite sides of the straight line* $ax + by + c = 0$, *according as the expressions* $ax_1 + by_1 + c$ *and* $ax_2 + by_2 + c$ *are of the same or opposite signs.*

Let P_1 and P_2 be the points (x_1, y_1) and (x_2, y_2) and AB the straight line



$$ax + by + c = 0. \quad \cdots \text{(i)}$$

Let the line P_1P_2 intersect AB at Q where the ratio $P_1Q : QP_2$ is $\lambda : 1$.

Then the co-ordinates of Q are

$$\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}.$$

As Q lies on AB, these co-ordinates must satisfy the equation (i).

Hence, $\quad a\dfrac{\lambda x_2 + x_1}{\lambda + 1} + b\dfrac{\lambda y_2 + y_1}{\lambda + 1} + c = 0,$

or, $\quad \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0$;

$\therefore \quad \lambda = -\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$

If P_1 and P_2 be on opposite sides of AB, Q must be within P_1P_2 and so the ratio $\lambda : 1$ is positive ; if P_1 and P_2 be on the same side of AB, Q must be external to AB and so $\lambda : 1$ is negative.

Hence, P_1 and P_2 will be on the same or positive sides of AB according as $-\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$ is negative or positive, *i.e.*, as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same or opposite signs.

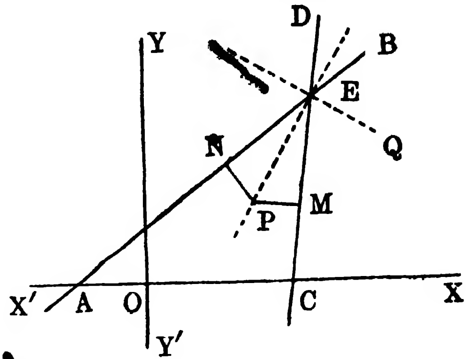
Cor. The point $P_1(x_1, y_1)$ is on the same side of the line $ax+by+c=0$ as the origin if ax_1+by_1+c is of the same sign as of c.

3·11. Equations of bisectors of the angles between two given straight lines

$$\mathbf{a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0.}$$

Any point x_1, y_1 (say) on any of the bisectors must be equi-

distant from the two lines, so that the perpendicular distances from it to the two lines must be equal in magnitude. If it be on the bisector of the angle in which the origin lies (as at P) both perpendiculars must be of the same sign, because the point and the origin lie on the same side with respect to each of the lines. On the other hand, if it be on the bisector of the angle in which the origin does not lie (as at Q) clearly it must be on the same side of the origin with respect to one of the lines (AB), and on opposite side of the origin with respect to the other line (CD). Hence, the two perpendiculars, being equal in magnitude, must be opposite in sign.



Hence, for points on one of the bisectors,

$$\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}$$

and for points on the other,

$$\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_2^2}} \quad \frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Hence, the equations to the bisectors of the angles between the two given lines (*i.e.*, the locus of x_1, y_1) are

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=\pm\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}}.$$

Note. If c_1 and c_2 be of the same sign, the + sign on the right-hand side will give the bisector of the angle in which the origin lies, and − sign will give the other bisector.

If c_1 and c_2 be of opposite signs, the − sign on the right-hand side will give the bisector of the angle containing the origin, and + sign will give the other bisector.

3·12. Illustrative Examples.

Ex. 1. *Find the perpendicular distance from the point* $(-2, -7)$ *to the straight line* $7y-24x=10$. *Is the point on the same side of the line as the origin, or on the opposite side ?*

The equation of the line can be written as

$$24x-7y+10=0.$$

The perpendicular distance from $(-2, -7)$ on it is given by

$$p=\frac{24.(-2)-7(-7)+10}{\pm\sqrt{24^2+7^2}}.$$

The proper sign to be chosen for the de[illegible] sign of the constant term $+10$).

Hence, $$p=\frac{-48+49+10}{+25}=\frac{11}{25}.$$

As $24(-2)-7(-7)+10$ (*i.e.*, $+11$) has the same sign as of the constant term $+10$, the point $(-2, -7)$ is on the same side of the given line as the origin.

The perpendicular from the point on the line is thus to be associated with positive sign, as has been found to be the case.

Ex. 2. *Find the bisector of the angle containing the origin between the lines* $4x-3y+5=0$ *and* $5x-12y-2=0$.

The bisectors of the angles between the given lines are [See § *3·11*],

$$\frac{4x-3y+5}{\sqrt{4^2+3^2}}=\pm\frac{5x-12y-2}{\sqrt{5^2+12^2}}.$$

Of these, (by *note of* § *3·11*), the $-$ sign on the right-hand will give the bisector of the angle containing the origin ($\because$ here 5 and -2 are of opposite signs).

Thus, the required bisector is

$$\frac{4x-3y+5}{5}=-\frac{5x-12y-2}{13},$$

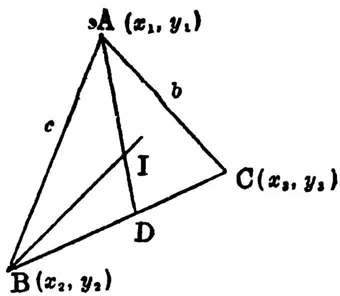
or, $77x-99y+55=0$, *i.e.*, $7x-9y+5=0$.

Ex. 3. *If* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *be the co-ordinates of the vertices*

of a triangle, and a, b, c respectively be the lengths of the sides opposite to these vertices, then the co-ordinates of the in-centre of the triangle are

$$\frac{ax_1+bx_2+cx_3}{a+b+c} \text{ and } \frac{ay_1+by_2+cy_3}{a+b+c}.$$

Let A, B, C denote the vertices having co ordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively; $BC=a$, $CA=b$, $AB=c$.



Let AD be the bisector of the angle BAC, meeting BC at D; then we know from geometry that

$$BD . DC=AB : AC=c . b.$$

$\therefore$ the co-ordinates of D are [See § 1·3]

$$\frac{cx_1+bx_2}{c+b}, \frac{cy_1+by_2}{c+b}.$$

Also, $\frac{BD}{DC}=\frac{c}{b}$. $\therefore \frac{BD}{BC}=\frac{c}{b+c}$. Hence, $BD=\frac{\quad}{b+c}$

Now, BI being the bisector of the angle ABC, meeting AD at I, I is the in centre of the triangle ABC.

Also, $\frac{DI}{IA}=\frac{BD}{BA}=\frac{ac}{b+c} : c=a . (b+c).$

Thus, the co ordinates of D and A being known, the co-ordinates of I are

$$\frac{a . x_1+(b+c) . \frac{cx_1+bx_2}{c+b}}{a+(b+c)} \text{ and } \frac{a . y_1+(b+c) . \frac{cy_1+by_2}{b+c}}{a+(b+c)}$$

i.e., $$\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}.$$

Examples III(b)

1. Find the perpendicular distance (i) from the point $(-1, -3)$ to the straight line $5x-12y+8=0$, (ii) from the point (a, b) to the straight line $\frac{x}{a}-\frac{y}{b}=1.$

2. Show that the distance of the point $(-5, 2)$ from the point $(-1, -1)$ is equal to its perpendicular distance from the straight line $4(y-1)=3(x-2)$.

3. Show that the point $(-3, 1)$ is equidistant from the two lines $4x-3y=2$ and $24x+7y=20$.

4. Find the perpendicular distance of the point of intersection of the two lines $2x+3y+6=0$ and $5x-y-19=0$ from the line $6x-8y=0$. What are the co-ordinates of the foot of the perpendicular ?

5. Find the points on the y-axis whose perpendicular distance from the straight line $5y=12x-16$ is 2.

6. Find the distance between the parallel lines $3x-4y+7=0$ and $3(x-2)=4y-3$.

7. (i) Show that the product of the perpendiculars from the two points $(\pm 4, 0)$ on the straight line $3x\cos\theta+5y\sin\theta=15$ is independent of θ.

(ii) Show that the product of the perpendiculars drawn from the two points $(\pm\sqrt{a^2-b^2}, 0)$ upon the straight line

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \text{ is } b^2.$$

8. (i) Find the perpendicular distance of the point $(2, 2)$ from the line joining the points $(-5, 1)$ and $(3, -5)$.

(ii) Prove that the points $(2, -6)$ and $(-4, -12)$ are equidistant from the line joining $(1, -4)$ and $(-3, -14)$, and are on opposite sides of it.

9. Show that the points $(2, -3)$ and $(-4, -2)$ are on opposite sides of the line $5x-7y-11=0$. Which of the points is on the same side of the line as the origin ?

10. Prove that the origin is inside the triangle whose vertices are $(2, 1)$, $(3, -2)$, $(-4, -1)$.

[*The origin lies on the same side with each vertex with respect to the line joining the other two.*]

11. Find the bisector of the angle between the lines $12x+5y=4$ and $3x+4y+7=0$ containing the origin.

12. Find the bisectors of the angles between the lines $24x-7y-2=0$ and $3y-4x+7=0$, and verify that these bisectors are at right angles. Which of these bisects the angle in which the origin lies ?

13. Find the in-centre of the triangle

(i) whose vertices are $(-1, -2)$, $(-1, 3)$ and $(11, -2)$.

[As in Ex. 3, § 3·12, let A, B, C denote the vertices having co-ordinates $(-1, -2)$, $(-1, 3)$, $(11, -2)$ respectively.

Then $AB=\sqrt{(-1+1)^2+(3+2)^2}=5$; $BC=\sqrt{(11+1)^2+(-2-3)^2}=13$

and $CA=\sqrt{(11+1)^2+(-2+2)^2}=12$.

Let AD be the bisector of the $\angle BAC$ meeting BC at D. Then $\dfrac{BD}{DC}=\dfrac{AB}{AC}=\dfrac{5}{12}$. $\therefore$ Co-ordinates of D $\left(\dfrac{5.11+12(-1)}{5+12}, \dfrac{5(-2)+12.3}{5+12}\right)$

i.e., $\left(\dfrac{43}{17}, \dfrac{26}{17}\right)$.

Also $\dfrac{BD}{DC}=\dfrac{5}{12}$. $\therefore$ $\dfrac{BD}{BC}=\dfrac{5}{17}$. $\therefore$ $BD=\dfrac{5}{17}\cdot BC=\dfrac{5}{17}\cdot 13=\dfrac{65}{17}$.

Now BI being the bisector of $\angle ABC$ meeting AD at I, I is the incentre of the $\triangle ABC$.

$\therefore$ $\dfrac{DI}{IA}=\dfrac{BD}{BA}=\dfrac{\frac{65}{17}}{5}=\dfrac{13}{17}$.

$\therefore$ Co-ordinates of I are $\left(\dfrac{13(-1)+17.\frac{43}{17}}{13+17}, \dfrac{13(-2)+17.\frac{26}{17}}{13+17}\right)$ *i.e.*, $(1, 0)$.]

(ii) whose sides are $x=3$, $y=-4$ and $3x+4y=17$.

[Solving the equations pair-wise let the co-ordinates of A B, C respectively are $(3, -4)$; $(11, -4)$; $(3, 2)$.

As in Ex. 13 (i), $AB=\sqrt{(11-3)^2+(-4+4)^2}=8$;

$BC=\sqrt{(11-3)^2+(-4-2)^2}=10$; $CA=\sqrt{(3-3)^2+(-4-2)^2}=6$.

$\dfrac{BD}{DC}=\dfrac{AB}{AC}=\dfrac{8}{6}=\dfrac{4}{3}$. $\therefore$ Co-ordinates of D are $\left(\dfrac{4.3+3.11}{4+3}, \dfrac{4.2+3(-4)}{4+3}\right)$

i.e., $\left(\dfrac{45}{7}, -\dfrac{4}{7}\right)$.

$\because$ $\dfrac{BD}{DC}=\dfrac{4}{3}$. $\therefore$ $\dfrac{BD}{BC}=\dfrac{4}{7}$. $\therefore$ $BD=\dfrac{4}{7}\cdot BC=\dfrac{4}{7}\cdot 10=\dfrac{40}{7}$.

Now, $\frac{DI}{IA} = \frac{BD}{BA} = \frac{\frac{40}{7}}{8} = \frac{5}{7}$.

$\therefore$ Co-ordinates of I are $\left(\frac{5.3+7.\frac{45}{7}}{5+7}, \frac{5(-4)+7(-\frac{4}{7})}{5+7}\right)$, *i.e.*, $(5, 2)$.]

14. The algebraic sum of the perpendicular distances on a straight line from the three given points $(5, -3)$, $(-2, 4)$ and $(-3, -7)$ is zero. Show that the straight line passes through a fixed point.

[Let the equation of the line be $ax+by+c=0$.

Then $\frac{5a-3b+c}{\pm\sqrt{a^2+b^2}} + \frac{-2a+4b+c}{\pm\sqrt{a^2+b^2}} + \frac{-3a-7b+c}{\pm\sqrt{a^2+b^2}} = 0$,

or, $5a-3b+c-2a+4b+c-3a-7b+c=0$, or, $-6b+3c=0$.

$$\frac{b}{c} = \frac{1}{2}.$$

Now $ax+by+c=0$ can always be written as $\frac{a}{c}x + \frac{b}{c}y+1=0$. But whatever be the value of a, the sum of perpendicular lengths remains unchanged.

Hence $a=0$ or $\frac{b}{c} = \frac{1}{2}$.

Therefore the required equation becomes $\frac{1}{2}y+1=0$ or $y=-2$.

Hence the straight line always passes through $(0, -2)$.

N. B. For example straight line passing through $(0, -2)$ is $y+2=m(x-0)$ or $mx-y-2=0$. And when we consider the sum of the $\perp$ length, it becomes $\frac{1}{\pm\sqrt{m^2+1}}(5m+3-2-2m-4-2-3m+7-2)=0$.]

ANSWERS

1. (i) 3. (ii) $\frac{ab}{\sqrt{a^2+b^2}}$. **4.** 5; (0, 0). **5.** (0, 2) and $\left(0, -\frac{42}{5}\right)$.

6. 2. **8.** (i) 5. **9.** $(-4, -2)$. **11.** $99x+77y+71=0$.

12. $44x-22y-37=0$, $4x+8y+33=0$; latter. **13.** (i) (1, 0). (ii) $(5, -2)$.

Higher Secondary Syllabus of Elective Mathematics :

CO-ORDINATE GEOMETRY

(Course for Class XI)

Circles, chords, tangents ; Normals and elementary properties connected with them ; Parabola, Ellipse, Hyperbola referred to their principal axes ; Analytical treatment of these curves in respect of (1) the focus and directrix properties, (2) tangents and normals and elementary properties connected with them, (3) centre and diameter.

[*Note*—Discussion should always be restricted to rectangular cartesian co-ordinates.]

CONTEN

IMPORTANT FORMULÆ & RESULTS

H. S. Co-ordinate Geometry (Class XI)

1. Equation of the circle

(i) standard form : $x^2+y^2=a^2$.

centre : $(0, 0)$; radius a.

(ii) general form : $x^2+y^2+2gx+2fy+c=0$

centre : $(-g, -f)$, radius $=\sqrt{g^2+f^2-c}$.

2. Circle with the given points (x_1, y_1) and (x_2, y_2) as extremities of a diameter $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

3. Equation of the tangent to the circle at (x_1, y_1)

(i) for standard form : $xx_1+yy_1=a^2$,

(ii) for general form :

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

4. Equation of the normal to the circle at (x_1, y_1)

(i) for standard form : $\dfrac{x}{x_1}=\dfrac{y}{y_1}$.

(ii) for general form : $x(y_1+f)-y(x_1+g)=fx_1-gy_1$.

5. Length of the chord of the circle $x^2+y^2=a^2$ intercepted by the line $y=mx+c$ is

$$2\frac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}.$$

6. Condition of tangency : condition that the line $y=mx+c$ may touch the circle $x^2+y^2=a^2$ is $c=\pm a\sqrt{1+m^2}$

$y=mx+a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$ for all values of m, and in that case the point of contact is

$$-\frac{am}{\sqrt{1+m^2}}; \quad \frac{a}{\sqrt{1+m^2}}$$

7. Length of the tangent from an external point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}.$$

8. Standard forms of the equations of conics.

(*a*) Parabola

(i) $y^2 = 4a(x - a)$ (with axis and directrix as axes of co-ordinates).

(ii) $y^2 = 4ax$. (standard form),
(with the vertex as origin and the axis and the tangent at the vertex as axes of co-ordinates).

(*b*) Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \textit{(standard form).}$$

(with centre as origin, and major and minor axes as axes of co-ordinates)

(*c*) Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(standard form)}$$

(with centre as origin and transverse and conjugate axes as axes of co-ordinates).

9. Parabola :

(i) Standard form $y^2 = 4ax$.

(ii) Latus rectum $= 4a$; focus is $(a, 0)$; extremities of the latus rectum are $(a, \pm 2a)$; directrix is $x = -a$.

(iii) Equation of the tangent at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

(iv) Normal at (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(v) Length of the chord intercepted by the straight line $y = mx + c$ is $\frac{4}{m^2}\sqrt{a(a - mc)(1 + m^2)}$.

(vi) Condition that $y = mx + c$ may touch the parabola is $c = \frac{a}{m}$ $(m \neq 0)$.

The line $y = mx + \frac{a}{m}$ is a tangent to the parabola for all values of m (except zero),

the point of contact being $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(vii) Parametric representation : $x = at^2$, $y = 2at$.

(viii) Equation of the diameter : $y = \frac{2a}{m}$.

10. Ellipse

(i) Standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(ii) Latus rectum $= 2a(1 - e^2) = 2\frac{b^2}{a}$.

(iii) Eccentricity : $b^2 = a^2(1 - e^2)$ or $e^2 = \frac{a^2 - b^2}{a^2}$.

(iv) Focal distances of $P(x_1, y_1)$:
$SP = a - ex_1$, $S'P = a + ex_1$; $SP + S'P = 2a$.

(v) Tangent at (x_1, y_1) : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(vi) Normal at (x_1, y_1) : $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}}$.

(vii) Length of the chord intercepted by the line $y = mx + c$ on the ellipse

$$= \frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}.$$

(viii) Condition of tangency :

The line $y = mx + c$ is a tangent to the ellipse if

$$c = \pm\sqrt{a^2m^2 + b^2}.$$

The line $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all values of m, and the point of contact is

$$-\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}.$$

(ix) Auxiliary circle : $x^2 + y^2 = a^2$.

(x) Parametric representation : $x = a \cos\theta$, $y = b \sin\theta$.

(xi) Diameter $y = -\frac{b^2}{a^2m}x$.

(xii) Director circle $x^2 + y^2 = a^2 + b^2$.

11. Hyperbola

(i) Standard equation : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(ii) Latus rectum : $= 2a(e^2 - 1) = 2\frac{b^2}{a}$.

(iii) Eccentricity : $b^2 = a^2(e^2 - 1)$ or $e^2 = \frac{a^2 + b^2}{a^2}$.

For rectangular (or equilateral) hyperpola
$a = b$; $e = \sqrt{2}$.

(iv) Focal distances of $P(x_1, y_1)$
$SP = ex_1 - a$, $S'P = ex_1 + a$,
$S'P - SP = 2a$.

(v) Equation of the tangent at (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(vi) Equation of the normal at (x_1, y_1) is

$$\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}.$$

(vii) Length of the chord of the hyperbola intercepted by $y = mx + c$ is

$$\frac{2ab\sqrt{1+m^2}\sqrt{c^2 - a^2m^2 + b^2}}{a^2m^2 - b^2}.$$

(viii) Condition of tangency :

The line $y = mx + c$ will be a tangent to the hyperbola if $c = \pm\sqrt{a^2m^2 - b^2}$.

The line $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola for all values of m, the point of contact being $\left(-\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$.

(ix) Equation of the diameter is $y = \frac{b^2}{a^2m}x$.

(x) Equation of the asymptotes : $y = \pm\frac{b}{a}x$.

CO-ORDINATE GEOMETRY

(To be taught in Class XI)

CHAPTER IV

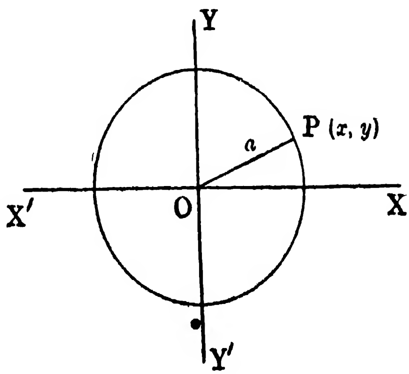
CIRCLE

4·1. Circle of given radius, with centre at the origin.

Let a be the radius of a circle whose centre is at the origin O. If P (x, y) be any point on the circle, the distance $OP = a$,

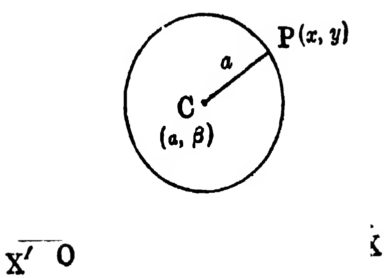
or, $OP^2 = a^2$;

$$\mathbf{x^2 + y^2 = a^2}$$



This being the relation satisfied by the co-ordinates (x, y) of any point on the circle, it represents the equation to the circle.

4·2. Circle of given radius, with centre at any given point.



Let C (α, β) be the centre, and a the radius of a circle. If P (x, y) be any point on the circle, the distance $CP = a$, or $CP^2 = a^2$;

$$\therefore \quad \mathbf{(x - \alpha)^2 + (y - \beta)^2 = a^2.}$$

This being the relation satisfied by the co-ordinates (x, y) of any point on the circle, it represents the equation to the circle.

Note. From above it is clear that any circle, with centre anywhere (say α, β) and radius of any length (a say) has equation of the form

$$x^2+y^2-2\alpha x-2\beta y+(\alpha^2+\beta^2-a^2)=0,$$

i.e., of the form $x^2+y^2+2gx+2fy+c=0$

where g, f, c are constants.

This is thus the most general form of the equation to a circle.

[See also § *4·3* and its note]

For a circle with centre as origin, g and f are zeroes.

4·3. *To show that the equation*

$$\mathbf{x^2+y^2+2gx+2fy+c=0}$$

always represents a circle (for all values of the constants g, f, c), and to find its centre and radius.

The given equation can be written in the form

$$x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c,$$

or, $(x+g)^2+(y+f)^2=g^2+f^2-c,$

or, $\{x-(-g)\}^2+\{y-(-f)\}^2=(\sqrt{g^2+f^2-c})^2.$

This shows that the distance of the variable point (x, y) from the fixed point $(-g, -f)$ is a constant $=\sqrt{g^2+f^2-c}$.

Hence, the locus represented by the equation is a circle, with *centre* $(-\mathbf{g}, -\mathbf{f})$, and *radius* $\sqrt{\mathbf{g^2+f^2-c}}$.

Note. The equation may be multiplied by a constant factor a, and put in the form

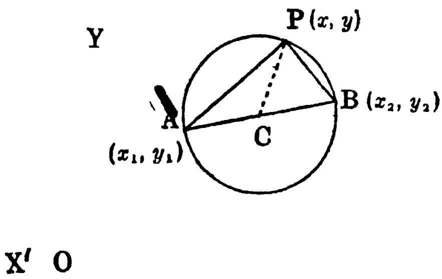
$$ax^2+ay^2+2g'x+2f'y+c'=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

This also represents a circle, but the centre here is not $(-g', -f')$, nor is the radius $=\sqrt{g'^2+f'^2-c'}$. In fact, whenever in a second degree equation (in rectangular co-ordinates) the coefficients of x^2 and y^2 are equal, and there is no term involving the product xy, the equation represents a circle. The equation (i) is thus the most general form of the equation of the circle. To get its centre and radius, we are to divide out the equation by the common coefficient a of x^2 or y^2, and reduce it to the form $x^2+y^2+2gx+2fy+c=0$.

Then $(-g, -f)$ are the co-ordinates of the centre and $\sqrt{g^2+f^2-c}$ is the length of the radius.

4·4. Circle with the given points (x_1, y_1) and (x_2, y_2) as extremities of a diameter.

$A(x_1, y_1)$ and $B(x_2, y_2)$ being the extremities of a diameter, the mid-point of AB, whose co-ordinates are $\frac{1}{2}(x_1+x_2)$, $\frac{1}{2}(y_1+y_2)$ is the centre. Also the radius $= \frac{1}{2}AB = \frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.



Hence, the equation to the circle is

$$\{x-\tfrac{1}{2}(x_1+x_2)\}^2+\{y-\tfrac{1}{2}(y_1+y_2)\}^2$$
$$=\tfrac{1}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\} \qquad \cdots \quad \text{(i)}$$

Alternatively

P being any point (x, y) on the circle, AB being a diameter, PA and PB must be at right angles. Now 'm' of PA is $\dfrac{y-y_1}{x-x_1}$ and that of PB is $\dfrac{y-y_2}{x-x_2}$. [See § *3·1(E)*]

Hence, for PA and PB to be at right angles,

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

or, $$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0} \qquad \text{(ii)}$$

which being the equation satisfied by the co-ordinates of any point P on the circle, it represents the required equation to the circle.

Note. It may be noted that the forms (i) and (ii) are identical, as can be shown by simplifying.

4·5. Circle passing through any three given points (x_1, y_1), (x_2, y_2), (x_3, y_3).

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (i)

be the equation to the circle.

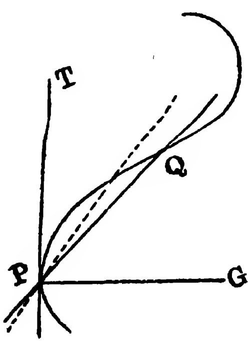
As it passes through the three given points,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$
$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0.$$

From these three equations, which are linear equations in the unknowns g, f, c, we get definite values of these unknowns when (x_1, y_1), (x_2, y_2) and (x_3, y_3) are given.

Substituting these values of g, f, c in (i), we get the required equation of the circle, as also its centre $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

4·6. Definitions of Tangent and Normal.



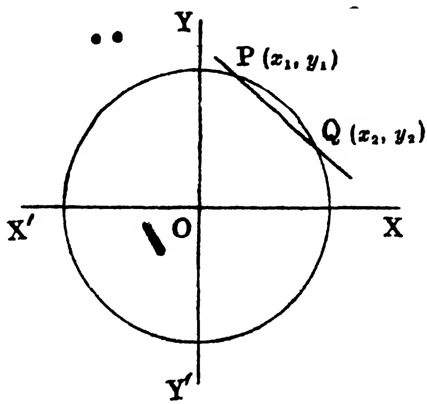
If P be any point on a curve, and if we take a neighbouring point Q on it, then if the line joining P and Q be turned about P so that the other *point of intersection* Q gradually approaches P, the limiting position PT of the line PQ, when Q ultimately coincides with P, is defined as the *tangent* to the curve at P.

The line PG through P, perpendicular to the tangent at P, is defined as the *normal* to the curve at P.

4·7. Equation to the tangent at a given point (x_1, y_1) on the circle :

(A) $x^2+y^2=a^2$.

(B) $x^2+y^2+2gx+2fy+c=0$.



(A) Let P be the point (x_1, y_1) on the circle

$$x^2+y^2=a^2, \qquad \text{(i)}$$

and let Q (x_2, y_2) be a neighbouring point on it.

The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \ldots \qquad \ldots \text{(ii)}$$

Now, since P and Q both lie on the circle (i), we have

$$x_1{}^2+y_1{}^2=a^2 \qquad \ldots \text{(ii)}$$

$$x_2{}^2+y_2{}^2=a^2 \qquad \ldots \text{(iv)}$$

$\therefore$ subtracting,

$$(x_2{}^2-x_1{}^2)+(y_2{}^2-y_1{}^2)=0,$$

whence $\dfrac{y_2-y_1}{x_2-x_1}=-\dfrac{x_2+x_1}{y_2+y_1}$.

$\therefore$ equation (ii) can be written as

$$y - y_1 = -\frac{x_2 + x_1}{y_2 + y_1}(x - x_1). \quad \dots \quad \dots \quad \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the chord PQ becomes the tangent at P, whose equation, [*from (v)*] then becomes

$$y - y_1 = -\frac{2x_1}{2y_1}(x - x_1),$$

or, $\quad x_1(x - x_1) + y_1(y - y_1) = 0,$

i.e., $\quad xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$ [*by (iii)*].

Hence, the equation of the tangent at (x_1, y_1) is

$$\mathbf{xx_1 + yy_1 = a^2.}$$

(B) Let P be the point (x_1, y_1) on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \dots \quad \dots \quad \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it. The equation t the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \dots \quad \dots \quad \text{(ii)}$$

Now, since P and Q both lie on the circle (i), we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots \quad \text{(iii)}$$

and $\quad x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \dots \quad \text{(iv)}$

$\therefore$ subtracting,

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0,$$

whence, $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0,$

or, $\quad \dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}.$

Hence, equation (ii) of chord PQ can be written as

$$y - y_1 = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}(x - x_1).$$

Now making Q approach P and ultimately coincide with it [so that (x_2, y_2) coincide with (x_1, y_1)], the chord becomes the tangent at P, whose equation is then

$$y - y_1 = -\frac{2(x_1 + g)}{2(y_1 + f)}(x - x_1),$$

or, $\quad (x_1 + g)(x - x_1) + (y_1 + f)(y - y_1) = 0,$

or, $\quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1,$

or, adding $gx_1 + fy_1 + c$ to both sides,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad [\text{ by (iii) }]$$

Hence, the tangent to the circle (i) at (x_1, y_1) is

$$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$$

4·8. Equation to the normal at $(\mathbf{x_1, y_1})$ to the circle :

(A) $\mathbf{x^2 + y^2 = a^2}$.

(B) $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$.

(A) The tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$, or, $y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1}$, of which the '$m$' is $-\frac{x_1}{y_1}$.

The normal, which is perpendicular to the tangent, through (x_1, y_1) is then

$$y - y_1 = \frac{y_1}{x_1}(x - x_1),$$

or, $\quad \frac{x - x_1}{x_1} = \frac{y - y_1}{y_1}$, or, $\frac{\mathbf{x}}{\mathbf{x_1}} = \frac{\mathbf{y}}{\mathbf{y_1}}$,

which evidently passes through the origin, *i.e.*, the centre of the circle.

(B) The tangent at (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

is $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$

i.e., $\quad x(x_1+g)+y(y_1+f)+(gx_1+fy_1+c)=0$

of which the 'm' is $-\dfrac{x_1+g}{y_1+f}$

The normal, which is perpendicular to the tangent, through (x_1, y_1) is then

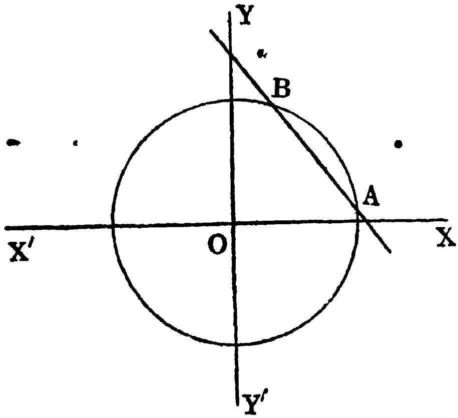
$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1),$$

or, $\quad \mathbf{x(y_1+f)-y(x_1+g)=fx_1-gy_1}.$

Note. Evidently this line passes through the point $(-g, -f)$, which is the centre of the circle.

Thus, *the normal at any point of a circle passes through its centre,* in other words, *the radius to any point of a circle is perpendicular to the tangent at the point.*

4·9. Length of the chord of the circle $\mathbf{x^2+y^2=a^2}$, intercepted by the straight line $\mathbf{y=mx+c}$.



At the points of intersection of the line with the circle both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$x^2+(mx+c)^2=a^2,$$

or, $\quad x^2(1+m^2)+2mcx+(c^2-a^2)=0, \quad \ldots \text{(i)}$

which being a quadratic equation in x, there are only two values of x and accordingly only two points of intersection of the straight line with the circle (which may be real and distinct, real and coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points A and B of intersection. Then x_1 and x_2 are the roots of (i).

$$\therefore \quad x_1 + x_2 = -\frac{2mc}{1+m^2} \text{ and } x_1 x_2 = \frac{c^2 - a^2}{1+m^2}.$$

$$\therefore \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$

$$= \frac{4m^2c^2}{(1+m^2)^2} - \frac{4(c^2 - a^2)}{1+m^2}$$

$$= \frac{4\{m^2c^2 - (c^2 - a^2)(1+m^2)\}}{(1+m^2)^2}$$

$$= \frac{4\{a^2(1+m^2) - c^2\}}{(1+m^2)^2}.$$

Again, $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$.

$$\therefore \quad y_1 - y_2 = m(x_1 - x_2).$$

Length of the chord AB

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2 (1+m^2)}$$

$$\sqrt{\frac{4\{a^2(1+m^2) - c^2\}}{1+m^2}} = 2\,\frac{\sqrt{\mathbf{a^2(1+m^2) - c^2}}}{\sqrt{\mathbf{1+m^2}}}.$$

Cor. Condition of tangency.

The given line will touch the circle if the chord intercepted is zero. Hence the condition that the given line $y = mx + c$ may touch the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1+m^2)$,

$$\text{or, } \quad \mathbf{c = \pm a\sqrt{1+m^2}}.$$

Alternative method for condition of tangency.

The perpendicular from the centre of the circle to the line is equal to the radius.

$\therefore$ perpendiculr from 0, 0 on $mx - y + c = 0$ is equal to a,

$$\text{or, } \quad \frac{c}{\pm\sqrt{1+m^2}} = a\,;$$

$$\therefore \quad c = \pm a\sqrt{1+m^2}.$$

Note. There are thus two tangents to the circle parallel to a given line $y = mx + c$, (*i.e.*, with a given m), namely.

$$y = mx \pm a\sqrt{1+m^2}.$$

4·10. *To show that $y=mx+a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$, and to find the point of contact.*

The tangent at (x_1, y_1) of the circle is

$$xx_1+yy_1=a^2 \quad \text{or} \quad xx_1+yy_1-a^2=0 \qquad \cdots \quad \text{(i)}$$

If the line $y=mx+a\sqrt{1+m^2}$ or $mx-y+a\sqrt{1+m^2}=0 \quad \cdots \quad \text{(ii)}$

be a tangent to the circle at (x_1, y_1), the equation (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{x_1}{m}=\frac{y_1}{-1}=\frac{-a^2}{a\sqrt{1+m^2}}=\frac{-a}{\sqrt{1+m^2}}.$$

$$\therefore \quad x_1=-\frac{am}{\sqrt{1+m^2}},\ y_1=\frac{a}{\sqrt{1+m^2}}.$$

The line (ii) therefore will touch the circle only if the assumed point (x_1, y_1) is really a point on the circle,

$$\text{i.e., if } \left(\frac{-am}{\sqrt{1+m^2}}\right)^2+\left(\frac{a}{\sqrt{1+m^2}}\right)^2=a^2$$

which is evidently satisfied.

Thus, $y=mx+a\sqrt{1+m^2}$ touches the circle, whatever m may be and the point of contact is given by

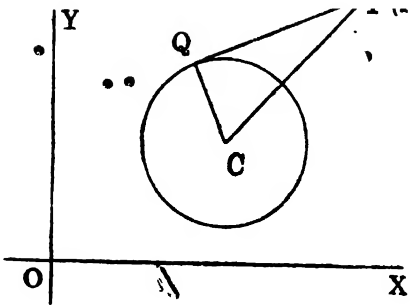
$$\mathbf{x_1=-\frac{am}{\sqrt{1+m^2}},\ y_1=\frac{a}{\sqrt{1+m^2}}}$$

Note. Similarly, $y=mx-a\sqrt{1+m^2}$ is also a tangent to the circle $x^2+y^2=a^2$, the point of contact being $\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}$.

4·11. *Length of the tangent from an external point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$.*

Let P be the point (x_1, y_1) and PQ a tangent to the circle from P. The centre C of the circle is $(-g, -f)$, and the radius CQ is $\sqrt{g^2+f^2-c}$.

Now since, as proved before (§ 4·8, *Note*), CQ is perpendicular to CP,



$$PQ^2 = CP^2 - CQ^2$$
$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

$$\therefore \quad PQ = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

Cor. The length of the tangent from the external point (x_1, y_1) to the circle $x^2+y^2=a^2$ is $\sqrt{x_1^2+y_1^2-a^2}$.

Note. We notice that when the co-ordinates (x_1, y_1) of any point are substituted for (x, y) on the left-hand side of the equation to a circle $x^2+y^2+2gx+2fy+c=0$ or $x^2+y^2-a^2=0$, we obtain the square on the length of the tangent from the point (x_1, y_1) to the circle. If this is positive, the point is outside the circle, and the length of the tangent is real. If the expression is negative, the point is inside the circle, and the length of the tangent is imaginary.

If the equation to the circle be in the form $ax^2+ay^2+2g'x+2f'y+c'=0$, we are first to divide out the equation by a, and reduce it to the form $x^2+y^2+2gx+2fy+c=0$ and then substitute (x_1, y_1) on the left side to get the square on the length of the tangent. [See § 4·3 *note* in this connection.]

4·12. Illustrative Examples.

Ex. 1. *Find the equation to the circle passing through the points* $(2, -3)$ *and* $(-3, -4)$ *and having its centre on the line* $7x+2y+6=0$.

Let (α, β) be the co-ordinates of the centre of the circle. As the circle passes through the points $(2, -3)$ and $(-3, -4)$, these points must be equidistant from the centre. Hence,

$$(\alpha-2)^2+(\beta+3)^2=(\alpha+3)^2+(\beta+4)^2,$$

or, $10\alpha+2\beta+12=0$, *i.e.*, $5\alpha+\beta+6=0$.

Also, since the centre lies on the given line, we have

$$7\alpha+2\beta+6=0. \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

From (i) and (ii), solving, $\alpha=-2$, $\beta=4$. The radius of the circle, r, is equal to the distance of the point $(2, -3)$ from the centre $(-2, 4.)$

$$\therefore \quad r^2=(2+2)^2+(-3\ -4)^2=65.$$

Hence, the required equation to the circle is

$$(x-\alpha)^2+(y-\beta)^2=r^2, \quad \text{or,} \quad (x+2)^2+(y-4)^2=65,$$

or, $x^2+y^2+4x-8y-45=0$.

Ex. 2. *Find the length of the chord intercepted by the straight line* $3x-4y+5=0$ *of the circle passing through the points* $(1, 2)$, $(3, -4)$ *and* $(5, -6)$.

Let $x^2+y^2+2gx+2fy+c=0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$

be the equation to the circle passing through the given points $(1, 2)$, $(3, -4$, and $(5, -6)$.

Then, substituting values of the co-ordinates,

$$5+2g+4f+c=0$$
$$25+6g-8f+c=0$$
$$\text{and} \quad 61+10g-12f+c=0$$

Solving these, we get $g=-11$, $f=-2$, $c=25$.

Hence, the circle (i) becomes

$$x^2+y^2-22x-4y+25=0 \quad \cdots \quad \cdots \quad \text{(ii)}$$

The given straight line is

$$3x-4y+5=0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

For the common points of intersection of (ii) and (iii), eliminating y, the abscissæ are the roots of

$$x^2+\left(\frac{3x+5}{4}\right)^2-22x-(3x+5)+25=0,$$

or, $5x^2-74x+69=0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$

If (x_1, y_1) and (x_2, y_2) be the co-ordinates of the points of intersection of (ii) and (iii), then x_1 and x_2 are roots of (iv).

$\therefore \quad x_1+x_2=\frac{74}{5},\ x_1x_2=\frac{69}{5}.$

$\therefore \quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=(\frac{74}{5})^2-4.\frac{69}{5}=\frac{4096}{25}.$

Also, (x_1, y_1) and (x_2, y_2) both lying on (iii),

$$3x_1-4y_1+5=0,\ 3x_2-4y_2+5=0.$$

$\therefore \quad 3(x_1-x_2)-4(y_1-y_2)=0. \qquad \therefore \quad (y_1-y_2)^2=\frac{9}{16}(x_1-x_2)^2.$

$\therefore$ the length of the intercepted chord being l,

$$l^2=(x_1-x_2)^2+(y_1-y_2)^2=(1+\tfrac{9}{16})(x_1-x_2)^2$$
$$=\tfrac{25}{16}\times\tfrac{4096}{25}=256.$$

$\therefore \quad l=16.$

Alternatively

The centre of the circle (ii) is (11, 2) and its radius r is $\sqrt{11^2+2^2-25}=10$ [See § *4·3*].

Perpendicular from this centre on line (iii) is

$$p=\frac{3.11-4.2+5}{\sqrt{3^2+4^2}}=6.$$

Now if AB be the chord along the line, and CN be the perpendicular on AB from the centre C, then N is the mid-point of AB. Also $AN^2=CA^2-CN^2$.

Hence, the length of the chord intercepted is

$$AB=2AN=2\sqrt{r^2-p^2}=2\sqrt{100-36}=16.$$

Ex. 3. *Show that the straight line* $4x+3y-31=0$ *touches the circle* $x^2+y^2-6x+4y=12$, *and find the point of contact.*

If possible, let (x_1, y_1) be the co-ordinates of the point on the circle

$$x^2+y^2-6x+4y-12=0 \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

at which the given line

$$4x+3y-31=0 \qquad \cdots \qquad \cdots \qquad \cdots \qquad \text{(ii)}$$

is a tangent.

As the equation to the tangent at (x_1, y_1) of the circle (i) is

$$xx_1+yy_1-3(x+x_1)+2(y+y_1)-12=0,$$

or, $\quad (x_1-3)x+(y_1+2)y-(3x_1-2y_1+12)=0$

this must be identical with (ii). Hence, comparing coefficients,

$$\frac{x_1-3}{4}=\frac{y_1+2}{3}=\frac{3x_1-2y_1+12}{31}$$

and each of these

$$=\frac{(3x_1-2y_1+12)-3(x_1-3)+2(y_1+2)}{31-3.4+2.3}=1$$

whence $\quad x_1=7,\ y_1=1.$

Substituting these values in (i), we see that the equation is satisfied.

Hence, there is the definite point (7, 1) on the circle (i), at which the tangent is identical with the line (ii).

Hence, (ii) touches (i), and the point of contact is (7, 1).

Alternatively, the centre of the circle (i) is clearly, (3, -2) and its radius is $\sqrt{(-3)^2+(2)^2-(-12)}=5.$

The straight line (ii) touches the circle (i) if the perpendicular distance from the centre on it is equal to the radius of the circle.

Now perpendicular from $(3,-2)$ on (ii) is

$$\frac{4.3+3.(-2)-31}{-\sqrt{4^2+3^2}}=5=\text{the radius of the circle.}$$

$\therefore$ (ii) touches (i).

Assuming the point of contact to be (x_1, y_1), comparing the equation of the tangent at the point with (ii), the co-ordinates x_1 and y_1 are obtained as before.

Ex. 4. *Prove that the locus of the middle points of any system of parallel chords of a circle is a diameter passing through the centre.*

Taking the centre of the circle as origin, the equation to the circle can be written as $x^2+y^2=a^2$ $\quad\cdots\quad\cdots\quad\cdots\quad$ (i)

Let any one of the system of parallel chords be

$$y=mx+c \quad\cdots\quad\cdots\quad\cdots\quad \text{(ii)}$$

where m is constant for all chords of the system, but c is different for different chord.

For points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the equation

$$x^2+(mx+c)^2=a^2,$$

or, $\quad x^2(1+m^2)+2mcx+(c^2-a^2)=0.$

Hence, if (x_1, y_1) and (x_2, y_2) be the extremities of the chord of intersection of (i) with (ii), x_1, x_2 being roots of the above equation.

$$x_1 + x_2 = -\frac{2mc}{1+m^2}.$$

Thus, X, Y being the middle point of the chord

$$X = \tfrac{1}{2}(x_1 + x_2) = -\frac{mc}{1+m^2}.$$

Also, X, Y being a point on (ii), $Y = mX + c$.

Eliminating c between these,

$$X = -\frac{m}{1+m^2}(Y - mX), \quad \text{or,} \quad X + mY = 0.$$

As this is free from c, this equation is satisfied by the co-ordinates of the middle point of every chord of the system, and it evidently represents the equation to a straight line passing through the origin *i.e.*, passing through the centre of the circle, and is thus a diameter.

Examples IV

1. Obtain the equation to a circle having its centre at (3, 7) and diameter 10.

What is the length of the intercept of this circle on the y-axis ? [*H. S. 1960, Compartmental*]

2. The extremities of a diameter of a circle have co-ordinates $(-4, 3)$ and $(12, -1)$. Find the equation to the circle. What length does it intercept on the y-axis ? [*H. S. 1961, Compartmental*]

3. Show that the equation $3x^2 + 3y^2 - 5x - 6y + 4 = 0$ represents a circle, and find its radius and co-ordinates of its centre.

points [illegible] and determine its centre and radius. [*H. S. 1961*]

5. Obtain the co-ordinates of the centre of the circle passing through the points $(1, 2)$, $(3, -4)$, $(5, -6)$, and determine the length of its diameter.

Is the origin inside, or outside the circle ? [*H. S. 1960*]

[Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Since (1, 2); (3, −4) and (5, −6) lie on circle, hence they satisfy the above equation.

$$\therefore \quad 2g+4f+c+5=0$$
$$6g-8f+c+25=0$$
$$10g-12f+c+61=0.$$

Solving the equation, $g=-11$; $f=-2$; $c=25$.

Co-ordinates of centre $(-g, -f)$ *i.e.*, (11, 2).

Radius $=\sqrt{g^2+f^2-c}=\sqrt{(-11)^2+(-2)^2-25}=10.$

$\therefore$ Diameter $=2\times 10=20.$

Length from origin to centre

$$=\sqrt{(-11)^2+(-2)^2}=\sqrt{125}=5\sqrt{5}>10 \text{ (radius of circle.}$$

Hence origin lies outside the circle.]

6. Find the equation to a circle which passes through the points $(0, -3)$ and $(3, -4)$, and which has its centre on the straight line $2x-5y+12=0$.

7. Find the equation to the circle passing through the origin and having intercepts 4 and -6 on the x-axis and y-axis respectively.

8. Find the equations to the circles which touch the axis of x and pass through the points $(1, -2)$ and $(3, -4)$.

9. A and B are two fixed points on a plane and the point P moves on the plane in such a way that $PA=2PB$ always. Prove analytically that the locus of P is a circle.

[*H. S. 1961, Compartmental*]

[x-axis is taken along AB with the origin at mid-point of AB and let $AB=2a$. So that the co-ordinates of A and B are $(-a, 0)$ and $(a, 0)$. And let the coordinates of P be (x_1, y_1).

Hence from the given condition

$$\sqrt{(x_1+a)^2+(y_1-0)^2}=2\sqrt{(x_1-a)^2+(y_1-0)^2}.$$

or $\quad (x_1+a)^2+y_1^2=4[(x_1-a)^2+y_1^2]$

or $\quad x_1^2+y_1^2-\frac{10}{3}ax_1+\frac{4}{3}a^2=0.$

Hence locus of P is $x^2+y^2-\frac{10}{3}ax+\frac{4}{3}a^2=0$ which is the equation of the circle with the centre as $(\frac{5}{3}a,\ 0)$ and

$$r=\sqrt{\left(\frac{5a}{3}\right)^2-\frac{y}{3}a^2}=\frac{\sqrt{13}}{3}a.\ ]$$

10. B, C are fixed points having co-ordinates $(3,\ 0)$ and $(-3,\ 0)$ respectively. If the vertical angle BAC be $90°$, show that the locus of the centroid of the triangle ABC is a circle whose equation you are to determine. [*H. S. 1961*]

11. (i) Find the length of the chord of the circle $x^2+y^2=64$, intercepted on the straight line $3x+4y-c=0$.

[As in § 4·12, $c\times-2$ for the common point of intersections eleminating y between $x^2+y^2=64$ and $3x+4y-c=0$, the abcissae are the roots of

$$x^2+\left(\frac{c-3x}{4}\right)^2=64$$

or $\qquad 25x^2-6cx+c^2-64\times16=0.$

If $(x_1,\ y_1)$ and $(x_2,\ y_2)$ be the co-ordinates of the point of intersection, then x_1, x_2 are the roots of the above equation.

$$\therefore\quad x_1+x_2=\frac{6c}{25}\text{ and }x_1x_2=\frac{c^2-64\times16}{25}$$

$$\therefore\quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{36c^2}{625}-\frac{4c^2-64\times64}{25}=\frac{64\times64\times25-64c^2}{625}.$$

Also $(x_1,\ y_1)$ and $(x_2,\ y_2)$ both on $3x+4y-c=0$,

$\therefore\quad 3y_1+4y_1-c=0$ and $3x_2+4y_2-c=0$

or $\quad 3(x_1-x_2)=-4(y_1-y_2).\qquad\therefore\quad (y_1-y_2)^2=\frac{9}{16}(x_1-x_2)^2.$

Now if length of the intercepted chord be 'l',

$$\therefore\quad l^2=(x_1-x_2)^2+(y_1-y_2)^2=\frac{25}{16}(x_1-x_2)^2$$

$$=\frac{25}{16}\cdot\frac{64\times64\times25-64c^2}{625}=\frac{64\times100-4c^2}{25}$$

$$=\frac{4}{25}(1600-c^2).\qquad\therefore\quad l=\frac{2}{5}\sqrt{1600-c^2}.]$$

(ii) Obtain the co-ordinates of the points of contact of any one of the two tangents to the above circle $x^2+y^2=64$, parallel to the line $3x+4y-c=0$. [*H. S. 1960*]

12. Prove that the straight line $y=x+a\sqrt{2}$ touches the circle $x^2+y^2=a^2$, and find its point of contact. [*H. S. 1961*]

13. Show that the line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y-12=0$, and find its point of contact.

14. Determine whether the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$. If it does, find the co-ordinates of the point of contact.

15. Find the equation to the circle

(i) having its centre at the point (3, 4), and touching the straight line $5x+12y+2=0$.

(ii) having its centre at $(1, -3)$ and touching the straight line $2x-y-4=0$.

16. Find the points at which the tangents to the circle $x^2+y^2-6x+8y=0$ is parallel to the line $3x+4y=0$.

17. Find the points on the circle $x^2+y^2-2x+6y-58=0$ at which the tangents are perpendicular to the line $4x-y=2$.

[Equation of any line perpendicular to

$$4x-y=2 \text{ is } x+4y=c \quad \cdots \quad \cdots \quad (1)$$

Eleminating x between (1) and $x^2+y^2-2x+6y-58=0 \quad \cdots \quad (2)$

we get $(c-4y)^2+y^2-2(c-4y)+6y-58=0$

or, $17y^2+2y(7-4c)+(c^2-2c-58)=0.$

If (1) touches (2), the two roots of above equations must be equal.

$\therefore$ $4(7-4c)^2=4.17.(c^2-2c-58)$ whence $c=23, -45$.

Let (x_1, y_1) be the point of contact of tangent. Equation of tangent a (x_1, y_1) of the circle (2) is

$$xx_1+yy_1-(x+x_1)+3(y+y_1)-58=0$$

or $$x(x_1-1)+y(y_1+3)-(x_1-3y_1+58)=0. \quad \cdots \quad (3)$$

If (1) touches (2), (1) and (3) must be identical,

$\therefore$ $\dfrac{x_1-1}{1}=\dfrac{y_1+3}{4}=\dfrac{x_1-3y_1+58}{c}.$

$\therefore$ solving and substituting the values of c,

$$x_1=3,\ y_1=5 \text{ and } x_1=-1,\ y_1=-1.$$

Hence, two points of contact are $(-1, -11)$ and $(3, 5)$.]

18. Show that the two circles

(i) $x^2+y^2+6x+14y+9=0$ and
$x^2+y^2-4x-10y-7=0$

touch each other externally.

[Co-ordinates of the circle are $(-3, -7)$ and radius $= \sqrt{(3)^2+(7)^2-9}=7=r_1$ of the circle $x^2+y^2+6x+14y+9=0$.

For the second circle $x^2+y^2+4x-10y-7=0$, co-ordinates of the centre $(2, 5)$, $r_2=\sqrt{(-2)^2+(-5)^2+7}=6$.

$\therefore \quad r_1+r_2=13.$

Distance between centres of two circles

$$=\sqrt{(-3-2)^2+(-7-5)^2}=13=r_1+r_2.$$

Hence two circles touch each other externally.]

(ii) $x^2+y^2-6x+6y-18=0$ and
$x^2+y^2-2y=0$

touch each other internally.

19. Find the length of the tangent drawn from

(i) the point $(-3, 11)$ to the circle
$$x^2+y^2-4x+2y-20=0.$$

[As in § 4·11, let bc be the length of tangent,

(i) $b=\sqrt{(-3)^2+(11)^2-4(-3)+2.11-20}=\sqrt{144}=12.$]

(ii) the point $(7, 2)$ to the circle
$$2x^2+2y^2+5x+y-15=0.$$

20. Show that the locus of the points from which the lengths of the tangents to the circles

$$x^2+y^2-3x+4y-7=0$$
and
$$x^2+y^2+2x-5y+1=0$$

are equal, is a straight line perpendicular to the line joining the centres of the circles.

[Let (x_1, y_1) be a point such that lengths from this to two given circles are equal.

Hence, $\sqrt{x_1^2+y_1^2-3x_1+4y_1-7}=\sqrt{x_1^2+y_1^2+2x_1-5y_1+1}$

or $\quad x_1^2+y_1^2-3x_1+4y_1-7=x_1^2+y_1^2+2x_1-5y_1+1$

or $\quad 5x_1-9y_1+8=0. \quad \dots \quad \dots \quad (1)$

Hence locus of (x_1, y_1) is $5x-9y+8=0$ which is a straight line and 'm' of this line say $m_1=\frac{5}{9}$.

Centre of $x^2+y^2-3x+4y-7=0$ is $(\frac{3}{2}, -2)$.

Centre of $x^2+y^2+2x-5y+1=0$ is $(-1, \frac{5}{2})$.

Equation of straight line passing through $(\frac{3}{2}, -2)$ and $(-1, \frac{5}{2})$ is

$$\frac{x-\frac{3}{2}}{-1-\frac{3}{2}}=\frac{y+2}{\frac{5}{2}+2} \text{ or } \frac{x-\frac{3}{2}}{-\frac{5}{2}}=\frac{y+2}{\frac{9}{2}} \quad \dots \quad (2)$$

'm' of this line say, $m_2=-\frac{9}{5}$.

Since, $m_1m_2=\frac{5}{9}\times-\frac{9}{5}=-1$, hence (1) and (2) are perpendicular.]

ANSWERS

1. $x^2+y^2-6x-14y+33=0$; 8.
2. $x^2+y^2-8x-2y-51=0$; $4\sqrt{13}$.
3. $\frac{1}{6}\sqrt{13}$; $(\frac{5}{6}, 1)$.
4. $x^2+y^2-6x+2y-15=0$; $(3, -1)$; 5.
5. $(11, 2)$; 20 ; outside.
6. $x^2+y^2-8x-8y-33=0$.
7. $x^2+y^2-4x+6y=0$.
8. $x^2+y^2-6x+4y+9=0$, $x^2+y^2+10x+20y+25=0$.

10. $x^2+y^2=1$.
11. (i) $\frac{2}{5}\sqrt{1600-c^2}$. (ii) $(\frac{24}{5}, \frac{32}{5})$ or $(-\frac{24}{5}, -\frac{32}{5})$.
12. $\left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.
13. $(-1, -1)$.
14. Yes ; $\left(1+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)$.
15. (i) $x^2+y^2-6x-8y=0$. (ii) $5x^2+5y^2-10x+30y+49=0$.
16. $(6, 0)$ and $(0, -8)$.
17. $(3, 5)$ and $(-1, -11)$.

19. (i) 12. (ii) 8.

CHAPTER V

CONICS

5·1. Definitions.

If a point moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the locus traced out by the point is defined to be a **conic.**

The fixed point is called the **focus** of the conic, and is usually denoted by the letter S.

The fixed straight line is referred to as the **directrix** of the conic.

The straight line through the focus perpendicular to the directrix is called the **axis.**

The constant ratio (of the distance of any point on a conic from the focus to the perpendicular distance of the point from the directrix) is called the **eccentricity** of the conic, and is usually denoted by the letter e.

When $e = 1$, the conic is defined to be a **parabola.**

When $e < 1$, the conic is defined to be an **ellipse.**

When $e > 1$, the conic is defined to be a **hyperbola.**

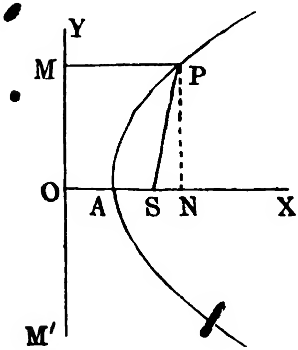
Note. The name 'Conic' or 'Conic-section' is due to the fact that these curves were first obtained and studied as sections of a cone by planes in various ways.

5·2. Parabola.

(A) *Equation with axis and directrix as axes of co-ordinates.*

Let S be the focus, and MM' the directrix of the parabola, which are fixed, and let OSX be the straight line through S perperpendicular to the directrix, so that it is the axis of the parabola, O being its point of intersection with the directrix.

Let us take OX as x-axis and OY (along the directrix) as y-axis, and let (x, y) be the co-ordinates of any point P on the



parabola. If PN and PM be perpendiculars from P on OX and OY, then $PM = ON = x$, $PN = y$.

Let the distance OS of S from the directrix be d. Then co-ordinates of S are $(d, 0)$.

From the definition of the parabola,

$$\frac{PS}{PM} = 1, \quad \text{or,} \quad PS = PM. \qquad PS^2 = PM^2$$

$$\text{or,} \quad (x-d)^2 + y^2 = x^2. \qquad \therefore \quad y^2 = 2d(x - \tfrac{1}{2}d)$$

or writing $d = 2a$, this can be written as

$$\mathbf{y^2 = 4a(x-a)} \qquad \cdots \qquad \cdots \quad \text{(i)}$$

If A be the middle point of OS, clearly $OA = AS = a$.

The co-ordinates of A are then $(a, 0)$ and they evidently satisfy the equation (i). Thus, A is a point on the parabola. This point A is called the **vertex** of the parabola.

(B) *Standard form of the equation to a parabola.*

If we transfer the origin to the vertex A, the equation (i) of the parabola reduces to

$$\mathbf{y^2 = 4ax} \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

which is the standard form of the equation to a parabola.

Here the vertex is the origin, the axis of the parabola is the x-axis, and the line through the vertex parallel to the directrix is the y-axis, a being the distance of the vertex from the focus, which is also equal to the perpendicular distance of the vertex from the directrix.

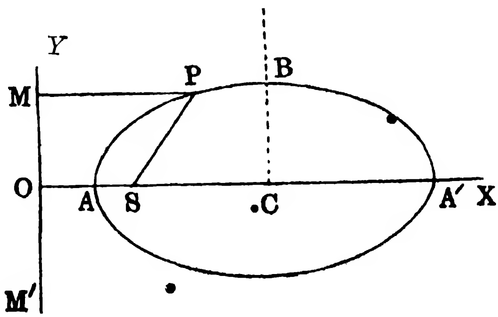
Note. For a discussion on the shape of the parabola and its elementary properties, see Chapter VI.

5·3. Ellipse.

(A) *Equation with directrix as y-axis and perpendicular to it through the focus as x-axis.*

Let S be the focus, MM' the directrix, and $e\,(<1)$ be the eccentricity of the ellipse.

SO being perpendicular on MM', OSX is taken as the x-axis, and OY (along the directrix) as y-axis. Let (x, y) denote the co-ordinates of any point P on the ellipse. Let the distance SO



(of focus from the directrix) be d, and let PM be the perpendicular from P on the directrix, so that $PM = x$.

Now, from the definition of the ellipse,

$$\frac{PS}{PM} = e, \quad \text{or,} \quad PS = e.PM.$$

$$\therefore \quad PS^2 = e^2.PM^2.$$

Hence, co-ordinates of S being evidently $(d, 0)$,

$$(x - d)^2 + y^2 = e^2x^2 \qquad \text{(i)}$$

This is then the equation to the ellipse with directrix as y-axis, and perpendicular to it through the focus as x-axis, d being the distance of the focus from the directrix.

(B) *Standard form of the equation to an ellipse.*

The above equation (i) can be written in the form

$$x^2(1-e^2)-2dx+d^2+y^2=0,$$

$$\text{or,}\quad (1-e^2)\left(x-\frac{d}{1-e^2}\right)^2+y^2=\frac{d^2}{1-e^2}-d^2=\frac{d^2e^2}{1-e^2},$$

$$\text{or,}\quad \left(x-\frac{d}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\left(\frac{de}{1-e^2}\right)^2.$$

Writing $\dfrac{de}{1-e^2}=a$, and transferring the origin to C whose co-ordinates are $\left(\dfrac{d}{1-e^2}, 0\right)$ *i.e.*, $\left(\dfrac{a}{e}, 0\right)$, (the axes remaining parallel to their original directions), the equation to the ellipse reduces to its standard form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

$$\textit{i.e.,}\qquad \frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1} \qquad \ldots \qquad \ldots \text{(ii)}$$

where $b^2=a^2(1-e^2)$.

Here the origin C is on the axis perpendicular to the directrix through the focus, at a distance $\dfrac{d}{1-e^2}=\dfrac{a}{e}$ from the directrix, and the y-axis, CB is parallel to the directrix.

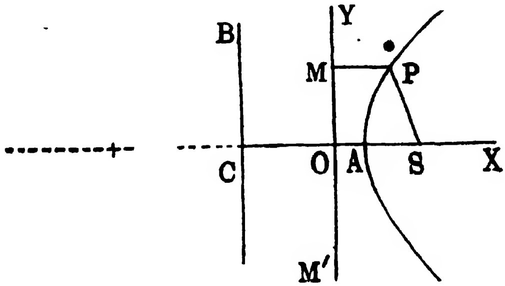
The point C is called the **centre** of the ellipse, the reason for which will be explained in Chapter VII.

$$\text{Distance, } CS=\frac{d}{1-e^2}-d=\frac{e^2d}{1-e^2}=ae.$$

Note. For a discussion on the shape of the ellipse, and its elementary roperties, see Chapter VII.

5·4. Hyperbola.

(A) *Equation with directrix as y-axis, and perpendicular to it through the focus as x-axis.*



Let S be the focus, MM' the directrix, and $e\,(>1)$ be the eccentricity of the hyperbola.

SO being perpendicular on MM', OSX is taken as the x-axis, and OY (along the directix) as y-axis. Let (x, y) denote the co-ordinates of any point P on the hyperbola. Let the distance SO (of focus from the directrix) be d, and let PM be the perpendicular from P on the directrix, so that $PM=x$.

Now, from the definition of the hyperbola,

$$\frac{PS}{PM}=e, \quad \text{or,} \quad PS=e.PM.$$

$$\therefore \quad PS^2=e^2.PM^2.$$

Hence, co-ordinates of S being evidently $(d, 0)$,

$$(x-d)^2+y^2=e^2x^2 \qquad \ldots \qquad \text{(i)}$$

This is then the equation to the hyperbola with directrix as y-axis, and perpendicular to it through the focus as x-axis, d being the distance of the focus from the directrix.

(B) *Standard form of the equation to a hyperbola.*

The above equation (i) can be written in the form (e being greater than 1 here),

$$x^2(e^2-1)+2dx-y^2=d^2\ ;$$

$$\text{or,}\quad (e^2-1)\left(x+\frac{d}{e^2-1}\right)^2-y^2=d^2+\frac{d^2}{e^2-1}=\frac{e^2d^2}{e^2-1}$$

$$\text{or,}\quad \left(x+\frac{d}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\left(\frac{de}{e^2-1}\right)^2.$$

Writing $\frac{de}{e^2-1}=a$, and transferring the origin to C whose co-ordinates are $\left(-\frac{d}{e^2-1},\ 0\right)$ *i.e.*, $\left(-\frac{a}{e},\ 0\right)$, (the axes remaining parallel to their original directions) the equation to the hyperbola reduces to its standard form

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

$$\text{or,}\quad \mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

where $b^2=a^2(e^2-1)$.

Here the origin C is on the axis perpendicular through the focus to the directrix, at a distance $\frac{d}{e^2-1}=\frac{a}{e}$ from the directrix on the side opposite to the focus. This point C is called the *centre* of the hyperbola.

Distance, $CS=d+\frac{d}{e^2-1}=\frac{de^2}{e^2-1}=ae.$

Note. For a discussion on the shape of the hyperbola, as also its lementary properties, see Chapter VIII.

5·5. Examples.

Ex. 1. *Find out the equation to the parabola whose focus is* $(-3, 4)$ *and irectrix is* $6x-7y+5=0$. [*H. S.* 1961]

Let (x_1, y_1) be the co-ordinates of any point on the parabola.

Its distance from the given focus $(-3, 4)$ is $\sqrt{(x_1+3)^2+(y_1-4)^2}$, and its erpendicular distance from the given directrix $6x-7y+5=0$ is $\frac{6x_1-7y_1+5}{\sqrt{6^2+7^2}}$.

'or the parabola these two distances are equal.

Hence, $(x_1+3)^2+(y_1-4)^2=\frac{(6x_1-7y_1+5)^2}{6^2+7^2}$. Thus, the co-ordinates (x_1, y_1) of any point on the parabola satisfy the equation

$$85\{(x+3)^2+(y-4)^2\}=(6x-7y+5)^2,$$

or, $$49x^2+84xy+36y^2+450x-610y+2100=0$$

which is then the required equation to the parabola.

Ex. 2. *Find the equation to the ellipse, whose focus is the point* $(-1, 1)$, *and directrix is the line* $x-y+3=0$, *and whose eccentricity is* $\frac{1}{2}$.

Let (x_1, y_1) be the co-ordinates of any point on the ellipse.

Its distance from the given focus $(-1, 1)$ is $\sqrt{(x_1+1)^2+(y_1-1)^2}$, and its perpendicular distance from the given directrix $x-y+3=0$ is $\frac{x_1-y_1+3}{\sqrt{1+1}}$. The ratio of these two distances is equal to the given eccentricity $\frac{1}{2}$ for any point on the ellipse.

Hence, $$\sqrt{(x_1+1)^2+(y_1-1)^2}=\frac{1}{2}\,\frac{x_1-y_1+3}{\sqrt{2}},$$

or, $$8\{(x_1+1)^2+(y_1-1)^2\}=(x_1-y_1+3)^2.$$

Thus, the co-ordinates (x_1, y_1) of the any point on the ellipse satisfy the equation

$$8\{(x+1)^2+(y-1)^2\}=(x-y+3)^2,$$

or, $$7x^2+2xy+7y^2+10x-10y+7=0,$$

which is then the required equation of the ellipse in question.

CHAPTER VI

PARABOLA

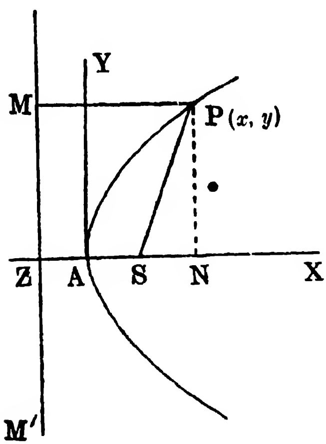
6·1. Parabola.

As has been defined in the previous chapter, a *parabola* is a curve traced out by a point which moves on a plane such that its distance from a fixed point on the plane is always equal to its perpendicular distance from a fixed straight line on that plane.

The fixed point is called the *focus*, and the fixed straight line is called the *directrix*.

6·2. Standard equation of a parabola.

Let S be the focus, and MM' the directrix of the parabola. Draw SZ perpendicular from S on MM', and let A be the middle point of ZS. Clearly, $\because$ $AS = AZ$, the point A is a point on the parabola. This point A is called the *vertex* of the parabola.



Let AS (the distance of the vertex from the focus) $= a$. Then AZ is also $= a$, and $SZ = 2a$.

Take A as origin, and ASX (perpendicular to the directrix through S) as x-axis, AY parallel to the directrix through A being the y-axis. Clearly the co-ordinates of S are $(a, 0)$.

P being any point on the parabola whose co-ordinates are (x, y), if PN be perpendicular on AX, and PM perpendicular to the directrix,

then $\quad PM = ZN = AZ + AN = a + x.$

Now, from the definition of a parabola,

$$PS = PM, \quad \text{or,} \quad PS^2 = PM^2.$$

$$\therefore \quad (x-a)^2 + y^2 = (a+x)^2.$$

$$\therefore \quad \mathbf{y^2 = 4ax.} \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

This being the relation satisfied by the co-ordinates of any point on the parabola, it represents the equation to the parabola in standard form with vertex as origin.

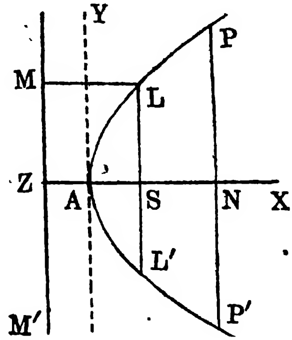
Here *a represents the distance of the vertex from the focus* (or from the directrix, the two being equal).

The line AX (perpendicular to the directrix through S), which is chosen as the axis of x here, is referred to as the *axis* of the parabola.

Note. In the previous chapter the equation was obtained with Z as origin, and then by transference of the origin to A, the equation to the parabola was obtained in the above standard form.

6·3. Shape and elementary properties of the parabola.

From the equation $y^2 = 4ax$, it is evident that if x be negative, y^2 being negative, y is imaginary. Hence, there is no real point with x negative, that is, there is no point of the parabola to the left of the origin A.



Again, for every positive value of x, there are two equal and opposite values of y. Hence corresponding to a point P on the parabola with positive ordinate PN, there is a point P' on the parabola with the same x $(=AN)$ with equal negative ordinate $P'N$. In other words, the parabola is symmetrical with respect to the x-axis AX, which bisects every chord of the type PNP' perpendicular to it. As x diminishes, and ultimately becomes zero, the two ordinates, which are equal in value and opposite in sign, become zero, and the point coincides with the origin A, which is the vertex of the parabola. As x becomes larger and larger, the values of y also become larger in magnitude. Hence, the shape of the parabola is as shown in the figure, closed at the left end A, and open on the right, y gradually becoming numerically larger and larger with x, *the whole curve being symmetrical about OX.*

It is for this property that OX is defined as the axis, and A is called the vertex.

A chord PNP' perpendicular to the axis (*i.e.*, parallel to the directrix) and bisected by the axis, is called a *double ordinate.* PN or $P'N$ is the ordinate of P or P'.

The chord LSL' through the focus S parallel to the directrix (and so perpendicular to the axis) is called the **Latus Rectum.**

If LM be the perpendicular on the directrix from the extremity L of the latus rectum, from the property of the parabola, $LS = LM = ZS = 2AS = 2a$.

Thus, the **Latus rectum = 4a,**

i.e., the *latus rectum is four times the distance of the focus from the vertex*, or, double the distance of the focus from the directrix.

The equation $y^2 = 4ax$ asserts the geometrical property of the parabola, that $PN^2 = 4AS.AN$, or, *the square on the ordinate is equal to the rectangle contained by the abscissa and the latus rectum.*

To sum up, we note that for the standard equation $y^2 = 4ax$ of the parabola,

(i) the vertex is the origin ;

(ii) the length of the latus rectum is $4a$;

(iii) the focus has co-ordinates $(a, 0)$;

(iv) the directrix has equation $x = -a$;

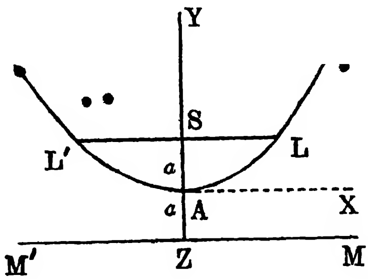
(v) the axis is the axis of x ;

and (vi) the co-ordinates of the extremities of the latus rectum are $(a, 2a)$ for L, and $(a, -2a)$ for L'.

Note. Equations $\mathbf{x^2=4ay}$, $\mathbf{y^2=-4ax}$, $\mathbf{x^2=-4ay}$.

If in the equation $y^2 = 4ax$ of a parabola, the axes of x and y are interchanged, *i.e.*, choosing the vertex A as origin, and latus rectum being $4a$ as before, the axis of the parabola (the line perpendicular to the directrix through the focus) be taken along the y-axis, the x-axis being parallel to the directrix, the equation to the parabola becomes $x^2 = 4ay$ and the figure is as shown here.

Here, the co-ordinates of the focus are $(0, a)$ and equation to the directrix is $y=-a$. The co-ordinates of the extremities L and L' of the latus rectum are $(2a, a)$ and $(-2a, a)$ respectively.

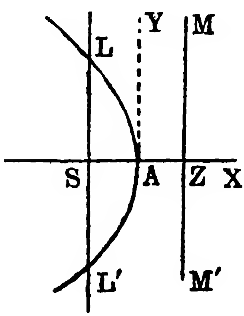


$x^2=4ay$

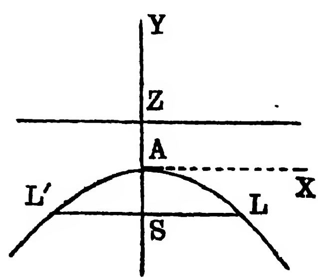
If in the equation $y^2=4ax$, the direction of the x-axis is reversed, *i.e.*, if the positive direction of the x-axis be taken from the vertex towards the directrix (the direction from the vertex to the focus being negative), the equation becomes $y^2=-4ax$, and the figure is as shown below, the concavity of he parabola being towards the negative side of the x-axis.

The co-ordinates of the focus are $(-a, 0)$ and directrix is $x=a$.

Similarly, in the equation $x^2=4ay$, if the direction of y-axis is reversed, the equation becomes $x^2=-4ay$, and the figure is as shown below, the concavity of the parabola being towards the negative side of the y-axis.



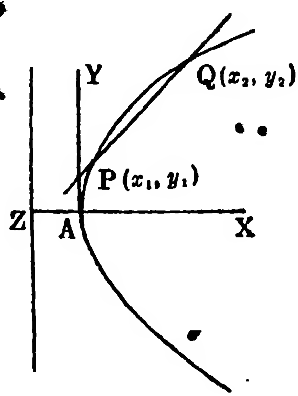
$y^2=-4ax$



$x^2=-4ay$

The co-ordinates of the focus are $(0, -a)$ and directrix is $y=a$.

6·4. Equation to the tangent at a given point (x_1, y_1) on the parabola $y^2 = 4ax$.



Let P be the point (x_1, y_1) on the parabola $y^2 = 4ax$ (i)

and let $Q\,(x_2, y_2)$ be a neighbouring point on it.

The equation to the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \ldots \qquad \text{(ii)}$$

Now since P and Q both lie on the parabola (i),

we have $y_1^2 = 4ax_1$ ... (iii) and $y_2^2 = 4ax_2$ (iv)

$\therefore$ substituting, $y_2^2 - y_1^2 = 4a(x_2 - x_1)$,

or, $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}.$$

equation (ii) can be written as

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \qquad \ldots \qquad \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the straight line PQ becomes the tangent at P, whose equation [*from (v)*] then becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1), \quad \text{or,} \quad yy_1 - y_1^2 = 2a(x - x_1),$$

i.e., $$yy_1 = y_1^2 + 2a(x - x_1) = 4ax_1 + 2a(x - x_1) \ [\ \textit{by (iii)}\ ].$$

Hence, the equation to the tangent at (x_1, y_1) is

$$\mathbf{yy_1 = 2a(x + x_1).}$$

Cor. *The tangent at the vertex of the parabola $y^2 = 4ax$ is the y-axis.*

6·5. Equation to the normal at (x_1, y_1) to the parabola $y^2 = 4ax$.

The tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

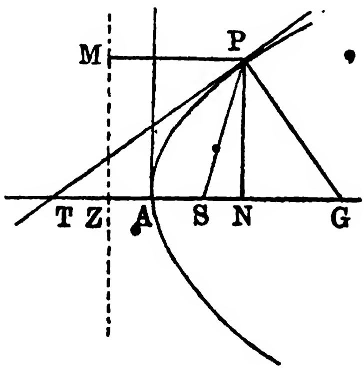
$$yy_1 = 2a(x + x_1), \text{ or, } y = \frac{2a}{y_1}(x + x_1),$$

of which the 'm' is $\frac{2a}{y_1}$.

The normal, which is perpendicular to the tangent through (x_1, y_1) is then

$$\mathbf{y - y_1 = -\frac{y_1}{2a}(x - x_1).}$$

6·6. Tangent and normal properties : Subtangent and Subnormal.



The length of the axis intercepted between the tangent and the foot of the ordinate of any point on the parabola is defined as the *subtangent* of the point.

The length of the axis intercepted between the normal and the foot of the ordinate of any point on the parabola is defined as the *subnormal* of the point.

Thus, if PT and PG be the tangent and normal at P, intersecting the axis at T and G respectively, and PN be the ordinate of P, then

TN is the subtangent of P and NG is the subnormal.

Now from the equation $yy_1 = 2a(x+x_1)$ of the tangent at $P(x_1, y_1)$, the point T where it intersects the axis is obtained by putting $y=0$, and thus for T, $x+x_1=0$ *i.e.*, $x=-x_1$.

Hence, $AT=AN$ in magnitude, T being on the negative side of A.

Hence, we get the geometrical property of the parabola that *the subtangent of any point on a parabola is bisected at the vertex.*

Again, in the equation $y-y_1 = -\dfrac{y_1}{2a}(x-x_1)$ of the normal at P, putting $y=0$, we get for the point G, $x-x_1=2a$, *i.e.*, $AG-AN=2a$, or $NG=2a=$ half the latus rectum. Hence, *the subnormal of any point of a parabola is constant and is equal to the semi-latus rectum.*

Further, $\because$ $AT=AN$ and $AS=AZ$, by adding we have $TS=ZN=PM$ (where PM is the perpendicular on the directrix) $=SP$. Hence, $\angle SPT=\angle PTS=$ the alternate $\angle TPM$. Also, $\because$ $\angle TPG$ is a right angle, $\angle SPG=\angle SGP$. Thus, we get further geometrical properties of the parabola, that :

(i) *the tangent at any point on a parabola bisects the angle between the focal distance of the point, and the perpendicular from the point to the directrix.*

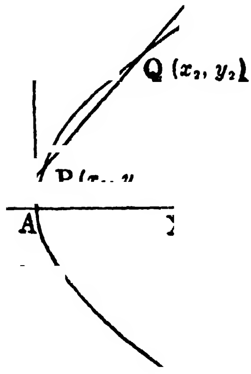
(ii) *the tangent at any point on a parabola makes equal angles with the focal distance of the point and the axis.*

(iii) *the normal at any point on a parabola is equally inclined to the focal distance of the point and the axis.*

6·7. Length of the chord of the parabola $y^2=4ax$, intercepted by the straight line $y=mx+c$.

At the points of intersection of the line with the parabola, both the equations are satisfied. Hence, eliminating y between

the two equations, the abscissæ of the points of intersection



$$(mx + c)^2 = 4ax,$$

$$\text{or,} \quad m^2x^2 + 2(mc - 2a)x + c^2 = 0, \qquad \cdots \quad \text{(i)}$$

which being a quadratic equation in x, there are only two values of x and accordingly only two points of intersection of the straight line with the parabola (which may be real and distinct, real and coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then x_1 and x_2 are roots of (i).

$$\therefore \quad x_1 + x_2 = -\frac{2(mc - 2a)}{m^2}, \quad \text{and} \quad x_1x_2 = \frac{c^2}{m^2}.$$

$$\therefore \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$

$$= \frac{4(mc - 2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16(a^2 - mca)}{m^4}.$$

Again, P and Q being on the given line,

$$y_1 = mx_1 + c, \quad y_2 = mx_2 + c.$$

$$\therefore \quad y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$ length of the chord PQ

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + m^2)}$$

$$= \sqrt{\frac{16(a^2 - mca)(1 + m^2)}{m^4}} = \frac{4}{m^2}\sqrt{a(a - mc)(1 + m^2)}$$

Cor. Condition of tangency.

The given line will touch the parabola only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y=mx+c$ may touch the parabola $y^2=4ax$ is

$$a-mc=0, \text{ or, } c=\frac{a}{m}.$$

6·8. *To show that* $y=mx+\dfrac{a}{m}$ *is a tangent to the parabola* $y^2=4ax$ *for all values of* m, *and to find the point of contact.*

The tangent at (x_1, y_1) of the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$, or, $y=\dfrac{2a}{y_1}x+\dfrac{2ax_1}{y_1}$ (i). If the line $y=mx+\dfrac{a}{m}$ (ii) be a tangent to the parabola at (x_1, y_1), the equation (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{2a}{y_1}=m, \quad \frac{2ax_1}{y_1}=\frac{a}{m}. \quad \therefore \quad x_1=\frac{a}{m^2}, \quad y_1=\frac{2a}{m}.$$

The line (ii) therefore will touch the parabola only if the assumed point (x_1, y_1) is really a point on the parabola $y^2=4ax$, *i.e.*, if $\left(\dfrac{2a}{m}\right)^2=4a\cdot\dfrac{a}{m^2}$ which is evidently satisfied.

Thus, $y=mx+\dfrac{a}{m}$ touches the parabola, whatever m may be, and the point of contact is given by

$$\mathbf{x_1=\frac{a}{m^2}, \quad y_1=\frac{2a}{m}.}$$

6·9. Co-ordinates of a point on the parabola $y^2=4ax$ expressed in terms of a single variable t.

We notice that if we substitute $x=at^2$, $y=2at$ in the equation $y^2=4ax$ of the parabola, the equation is automatically satisfied for all values of t. Hence, any point on the parabola can have its two co-ordinates expressed in terms of a single variable t in the form

$$\mathbf{x=at^2, \quad y=2at.}$$

For different points, t will be different, and for a definite point on the parabola, t will be definite and unique. A point on the parabola will thus be referred to as the point t.

In working out many examples in parabola, when given by its standard form $y^2 = 4ax$, this assumption of the co-ordinates of a point on it in terms of the single variable t in the above form will be very helpful.

In this connection we may note that the equation to the tangent to the parabola $y^2 = 4ax$ at the point t is (See § 6·4)

$y.2at \quad 2a(x + at^2)$, or, $y = \dfrac{x}{t} + at$. Also the normal at the point t is (See § 6·5)

$$y - 2at = -\frac{2at}{2a}(x - at^2), \text{ or. } y + tx = 2at + at^3$$

Note. Significance of t.

From the equation of the tangent at t it is apparent that $\frac{1}{t}$ is the gradient of the tangent line at t, *i.e.*, t represents the cotangent of the angle made by the tangent line at t with the x-axis.

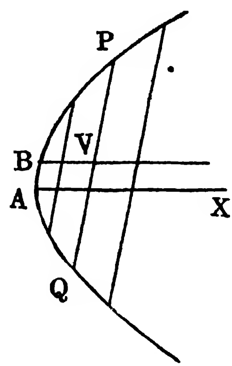
6·10. Locus of the middle points of a system of parallel chords ; diameter.

Let PQ, given by the equation

$$y = mx + c \qquad \ldots \quad \text{(i)}$$

be any one of a system of a parallel chords of the parabola

$$y^2 = 4ax \qquad \ldots \quad \text{(ii)}.$$



As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating x, the ordinates are given by the roots of the equation

$$y^2 = 4a\left(\frac{y - c}{m}\right),$$

or, $$my^2 - 4ay + 4ac = 0.$$

If (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points of intersection P and Q, we get $y_1 + y_2 = \frac{4a}{m}$. Hence, for the middle point V,

$$y = \tfrac{1}{2}(y_1 + y_2) = \frac{2a}{m}.$$

This being a relation free from c, it is satisfied by the middle point of every chord of the system.

Hence, this represents the locus of the middle points of the system of parallel chords, and we know that this represents a straight line parallel to the x-axis.

We thus see that *the locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to its axis.*

Such a straight line is defined to be a *diameter* of the parabola, bisecting the particular system of parallel chords.

For different m, *i.e.*, for differently directed systems of parallel chords we get different diameters.

Note. If B be the point where the diameter in equation meets the parabola, we have for B also, $y = \frac{2a}{m}$. Hence, $x = \frac{y^2}{4a} = \frac{a}{m^2}$. At this point then the tangent line is $y = mx + \frac{a}{m}$ [See § *6·8*], and this is also parallel to the system of chords bisected by the particular diameter. B is called the vertex of that diameter.

In fact any line parallel to the axis of the parabola is a diameter bisecting all chords parallel to the tangent at its extremity, *i.e.*, at the vertex of that diameter.

6·11. Illustrative Examples.

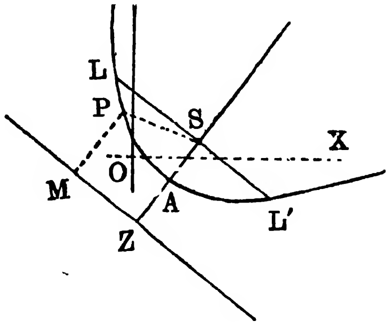
Ex. 1. *The focus of a parabola is* (6, 2) *and its vertex is* (3, −2). *Find the equation to the parabola, and the length of its latus rectum. Also obtain the co-ordinates of the extremities of its latus rectum.*

Let S be the focus having co-ordinates (6, 2), and A the vertex having co-ordinates (3, −2). The distance $AS = \sqrt{(6-3)^2 + (2+2)^2} = 5$.

Hence, the length of the latus rectum of the parabola $=4AS=20$.

Again, 'm' of the line joining A and S is $\frac{2-(-2)}{6-3}=\frac{4}{3}$, and this line AS is the axis of the parabola. The latus rectum is perpendicular to AS through S, and so its equation is

$$y-2=-\tfrac{3}{4}(x-6) \quad \dots \quad \text{(i)}$$



If (x_1, y_1) be the co-ordinates of an extremity (say L or L') of the latus rectum,

$\because$ $SL=$ semi-latus rectum $=10$,

$$(x_1-6)^2+(y_1-2)^2=100 \quad \text{(ii)}$$

and as (x_1, y_1) lies on (i),

$$y_1-2=-\tfrac{3}{4}(x_1-6) \quad \text{(iii)}$$

From (ii) and (iii),

$$(x_1-6)^2(1+\tfrac{9}{16})=100,$$

$\therefore$ $(x_1-6)^2=64$. $\therefore$ $x_1-6=\pm 8$.

Taking $+$sign, $x_1=14$, and from (iii), $y_1=-4$.

Taking $-$sign, $x_1=-2$, and from (iii), $y_1=8$.

Thus, the co-ordinates of the extremities of the latus rectum are $(14, -4)$ and $(-2, 8)$.

Lastly, produce SA to Z, such that $AZ=AS$. Then the co-ordinates of Z being (α, β) say, as A is the mid-point of ZS,

$$\tfrac{1}{2}(\alpha+6)=3 \quad \text{and} \quad \tfrac{1}{2}(\beta+2)=-2.$$

$\therefore$ $\alpha=0$, $\beta=-6$.

Now since the vertex is the mid-point of the perpendicular from the focus on the directrix, the point Z is clearly the foot of the perpendicular. The directrix of the parabola is therefore the line through Z perpendicular to ZAS, and hence its equation is

$$y+6=-\tfrac{3}{4}(x-0), \quad \text{or,} \quad 3x+4y+24=0. \quad \dots \quad \dots \quad \text{(iv)}$$

If (x, y) be the co-ordinates of any point P on the parabola,

$\because$ $PS=$ perp. distance from P on (iv)

$$\sqrt{(x-6)^2+(y-2)^2}=\frac{3x+4y+24}{\sqrt{3^2+4^2}}=\frac{3x+4y+24}{5},$$

whence $25\{(x-6)^2+(y-2)^2\}=(3x+4y+24)^2$

which is the equation to the parabola.

Ex. 2. *By suitably transferring the origin, show that the equation $3y^2-10x-12y-18=0$ reduces to the standard form of equation to a parabola, and hence obtain the co-ordinates of its vertex and focus, and the length of its latus rectum. Also determine the equation to its directrix.*

The given equation can be written as

$$3(y^2-4y)=10x+18, \quad \text{or,} \quad 3(y-2)^2=10(x+3).$$

Hence, transferring the origin to $(-3, 2)$, the equation reduces to the form

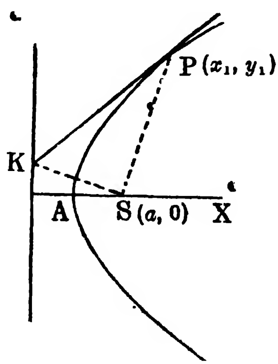
$$y^2=\tfrac{10}{3}x \qquad \ldots \qquad \ldots \qquad \ldots \quad \text{(i)}$$

which is the standard form of the equation to a parabola. Comparing this with the equation $y^2=4ax$, whose latus rectum is $4a$, vertex is the origin, focus is $(a, 0)$ and directrix is $x=-a$, we see that for the parabola (i), the latus rectum is $\frac{10}{3}$, the vertex is the new origin, and referred to this, the co-ordinates of the focus are $(\frac{5}{6}, 0)$, and the equation to the directrix is $x=-\frac{5}{6}$.

Hence, returning back to the given old origin, the co-ordinates of the vertex are $(-3, 2)$, the co-ordinates of the focus are $(-3+\frac{5}{6}, 2+0)$ *i.e.*, $(-2\frac{1}{6}, 2)$, and the equation to the directrix is $x=-\frac{5}{6}-3$ *i.e.*, $x=-3\frac{5}{6}$.

The latus rectum has already been shown to be $\frac{10}{3}$.

Ex. 3. *Prove that the length of any tangent to a parabola intercepted between its point of contact and the directrix subtends a right angle at the focus.*



Taking the vertex as origin and axis as x-axis, let the equation to the parabola be $y^2=4ax$. $\qquad \ldots \qquad \ldots \qquad \ldots \quad$ (i)

Then its focus S has co-ordinates $(a, 0)$ and the equation to the directrix is $x=-a$. $\qquad$ (ii)

The tangent at any point $P(x_1, y_1)$ is given by

$$yy_1 = 2a(x+x_1) \quad \cdots \quad \text{(iii)}$$

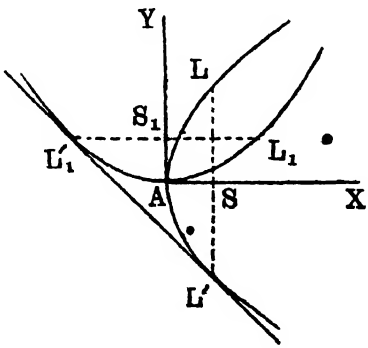
This meets the directrix (ii) at K, whose $x = -a$

and $y = \frac{2a}{y_1}(-a+x_1)$ [*from (iii)*].

Now the slope 'm' of the line $SP = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}$ and the slope 'm'' of the line SK is $\frac{\frac{2a}{y_1}(x_1 - a) - 0}{-a-a} = -\frac{x_1 - a}{y_1}$.

$\therefore \quad mm' = -1$. Hence, SP and SK are at right angles, *i.e.*, PK subtends a right angle at S.

Ex. 4. *Two equal parabolas have the same vertex, and their axes are at right angles; prove that their common tangent touches each at an end of its latus rectum.*



Let $y^2 = 4ax \quad \cdots$ (i) be the equation to one of the parabolas. The other parabola, which is equal to it (and hence has an equal latus rectum $4a$), having the same vertex A (chosen as origin) and having its axis perpendicular to that of (i) (*i.e.*, along the y-axis) is then given by

$$x^2 = 4ay \quad \cdots \quad \cdots \quad \text{(ii)}$$

Any tangent to (i) is $y = mx + \frac{a}{m}$ (iii)

having the point of contact at $\frac{a}{m^2}$, $2am$.

If it be a tangent to (ii) also, the two points of intersection of (ii) and (iii) must coincide, or eliminating y, the two roots of

$$x^2 - 4a\left(mx + \frac{a}{m}\right) = 0 \quad \cdots \quad \cdots \quad \cdots \text{(iv)}$$

are equal, which requires

$$(4am)^2 + 4.\frac{4a^2}{m} = 0,$$

or, $m^3 = -1$. $\therefore$ $m = -1$.

$\therefore$ the common tangent of (i) and (ii) is $y = -x - a$,

or, $x + y + a = 0$.

The point of contact of this common tangent on (i) is [putting $m = -1$ here] $(a, \ -2a)$, which is clearly the co-ordinates of one extremity L' of the latus rectum.

The point of contact of the common tangent on (ii) [$\because$ for the equal roots for (iv) the sum of the roots is $4am = -4a$] is given by $x = -2a$, and hence from (ii) $y = a$. But $(-2a, \ a)$ are clearly the co-ordinates of the extremity L'_1, of the latus rectum of (ii).

Hence, the common tangent of the two parabolas touch each at an end of its latus rectum.

Examples VI

1. Find the point on the parabola $y^2 = 18x$ at which the ordinate is three times the abscissa.

2. The parabola $y^2 = 4ax$ passes through the point $(2, \ -6)$. Find the length of its latus rectum.

3. Find the equation to the line joining the vertex to the positive end of the latus rectum of the parabola $y^2 = 8x$.

4. A double ordinate of the parabola $y^2 = 4ax$ is of length $8a$. Prove that the line joining the vertex to its two ends are at right angles. [*H. S. 1960*]

5. Find the latus rectum of the parabola whose focus is $(2, \ -3)$, and directrix is $5x - 12y + 6 = 0$.

6. Find the equation to the parabola

(i) whose focus is $(5, \ 3)$ and directrix is $3x - 4y + 1 = 0$.

(ii) whose focus is $(-6, \ -6)$ and vertex is $(-2, \ 2)$.

[Consider the figure of Ex. 1 of § 6·11.

Equation of the line joining $S(-6, -6)$ and $A(-2, 2)$ is

$$\frac{x+6}{-2+6}=\frac{y+6}{2+6} \quad \text{or,} \quad y+6=2(x+6).$$

Let the co-ordinates of z be (α, β).

$\therefore \quad \frac{\alpha-6}{2}=-2$ and $\frac{\beta-6}{2}=2$ [Since co-ordinates of focus $(-6, -6)$ and vertex $A(-2, 2)$]

$\therefore \quad \alpha=2, \ \beta=10.$

Equation of directrix which passes through $z(2, 10)$ and perpendicular to AS is line (1) is $y-10=-\frac{1}{2}(x-2)$, or, $x+2y-22=0$.

Let $P(x, y)$ any point on the parabola.

$\therefore \quad SP^2=(x+6)^2+(y+6)^2$ and $PM=\frac{x+2y-22}{\sqrt{1^2+2^2}}$.

Since, $SP^2=PM^2$. $\therefore \quad (x+6)^2+(y+6)^2=\frac{(x+2y-22)^2}{5}$ is the equation of parabola which on simplification becomes

$$4x^2+y^2-4xy+104x+148y-12y=0. \text{]}$$

7. Find the vertex, focus and latus rectum of each of the parabolas

(i) $y^2=4(x+y)$.

[$y^2=4(x+y)$ or $y^2-4x=4x$ or $(y-2)^2=4(x+1)$.

Replacing $y-2=Y$ and $x+1=X$, $Y^2=4X$ which is the equation of the parabola with vertex at $X=0$ and $Y=0$ *i.e.*, $x+1=0$ and $y-2=0$ *i.e.*, $x=-1$, $y=2$ and focus at $X=1$, $Y=0$ *i.e.*, $x+1=1$ and $y-2=0$, *i.e.*, $x=0$, $y=2$ and latus-rectus $=4$.]

(ii) $x^2+2y=8x-7$.

$x^2+2y=8x-7$ or $(x-4)^2=-2(y-\frac{9}{2})=2(\frac{9}{2}-y)$.

Replacing $x-4=X$, $\frac{9}{2}-y=Y$, equation becomes $X^2=2Y$ which is the equation of parabola with vertex at $X=0$, $Y=0$ *i.e.*, $x-4=0$ and $\frac{9}{2}-Y=0$ *i.e.*, $x=4$; $y=\frac{9}{2}$; focus at $X=x-4=0$ and $Y=\frac{9}{2}-y=\frac{1}{2}$. $\therefore$ $x=4$, $y=4$. and latus-rectum $=2$.]

8. Find out the equation of the tangent to the parabola $y^2=4ax$ at the extremity of the latus rectum. [*H. S. 1960*]

9. Find the equation to the tangent to the parabola

(i) $y^2 = 9x$ at the point whose ordinate is 6.

(ii) $y^2 = 12x$ at the positive extremity of the latus rectum.

[Equation of latus rectum is $x=3$, for the two extremities of latus-rectum, solve $y^2=12x$ and $x=3$. Co-ordinates of extremities are $(3, \pm 6)$. Equation of tangent at positive extremity $(3, 6)$ is $y.6=6(x+3)$ or $y=x+3$.]

10. Show that the foot of the perpendicular from the focus of the parabola $y^2=4ax$ on any tangent lies on the y-axis.

[*H. S. 1961, Compartmental*]

11. Prove that the tangents at the extremities of the latus rectum of a parabola meet on the directrix, and are at right angles.

12. The two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles. Find the locus of P.

13. (i) Prove that any two perpendicular tangents to the parabola $y^2=4ax$ intersect on the directrix.

(ii) If two tangents to a parabola are at right angles, show that their points of contact are at the extremities of a focal chord.

14. A tangent to the parabola $y^2=12x$ makes an angle 45° with the axis. Find the co-ordinates of its point of contact.

15. A tangent to the parabola $y^2=4ax$ makes an angle 60° with the axis. Find its point of contact.

16. Find the equation to the tangent to the parabola $y^2=7x$ which is parallel to the straight line $x-4y-3=0$. Find also its point of contact.

17. Find the equation of the tangent to the parabola $y^2=8x$ which is perpendicular to $x+2y+7=0$.

18. Find the point on the parabola $y^2=8x$ at which the normal is inclined at an angle 60° with the positive direction of the x-axis.

[Equation of normal at (x_1, y_1) to parabola $y^2=8x$ is

$$y-y_1=-\frac{y_1}{4}(x-x_1).$$

Since this line is inclined at an angle 60° with x-axis.

$\therefore\quad -\frac{y_1}{4}=\tan 60^\circ=\sqrt{3}.\qquad \therefore\quad y_1=-4\sqrt{3}.$

Again since (x_1, y_1) lies on parabola $y_1^2=8x_1$ i.e. $(-4\sqrt{3})^2=8x_1$.

$\therefore\quad x_1=6.$ Hence point of contact of normal is $(6, -4\sqrt{3})$.]

19. Find the equation to the locus of the foot of the perpendicular from the vertex on the tangent at any point of the parabola $y^2=4ax$.

[Equation of tangent at (x_1, y_1) the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$.

Equation of normal to above line passing through (0, 0) is $y=-\frac{y_1}{2a}x$.

Solving above two equations to foot of perpendicular is obtained and is given by

$$\frac{y}{2ax_1y_1}=\frac{x}{-4a^2x_1}=\frac{1}{4(x+a)}.$$

Locus of the foot of the perpendicular is obtained by elimination.

From above relation,

$$x(x^2+y^2)=-\frac{a^2x_1^3}{4(x_1+a)^2}=-ay^2 \quad [\text{ Substitute } y_1^2=4ax_1\ ]$$

or, $x(x^2+y^2)+ay^2=0$ which is the reqd locus.]

20. Find the equation to the chord of the parabola $y^2=8x$ which is bisected at the point $(2, -3)$.

[Let $y=mx+c$ be a chord to parabola $y^2=8x$.

Eliminating x between two equations, $my^2-8y+8c=0$.

If (x_1, y_1) and (x_2, y_2) are points of intersection of chord with parabola

$$y_1+y_2=\frac{8}{m}.\quad \text{But } \tfrac{1}{2}(y_1+y_2)=-3=\frac{8}{m}.\quad \therefore\quad m=-\tfrac{4}{3}.$$

Hence equation of chord is $y=-\frac{4}{3}x+c$ and passes through $(2, -3)$.

$\therefore\quad -3=-\frac{8}{3}-2+c.\qquad \therefore\quad c=-\frac{1}{3}.$

Hence equation of chord is $y=-\frac{4}{3}x-\frac{1}{3}$ or, $3y+4x+1=0$.]

21. Prove that the locus of the middle points of all chords of the parabola $y^2=4ax$ which are drawn through the vertex is the parabola $y^2=2ax$.

22. Find the length of the chord of the parabola $y^2=12x$ which is inclined at an angle of 45° with the axis, and passes through the point (1, 3).

23. Find the length of the chord of the parabola $y^2 = 20x$ along the straight line $x - 2y + 4 = 0$.

24. Find the length of the normal chord of the parabola $y^2 = 4ax$ through an extremity of the latus rectum.

25. Find the middle point of the line $3y - 4x = 4$ intercepted by the parabola $y^2 = 8x$.

[Let (h, k) be mid-point.

$\therefore \quad \frac{4}{\frac{4}{3}} = 3$ [Note § 6·10]

Also since (h, k) lies on $3y - 4x = 4$,

$\therefore \quad 3k - 4h = 4$ or, $4h = 3k - 4 = 9 - 4 = 5$. $\therefore \quad h = \frac{5}{4}$.

$\therefore$ Required middle-point $(\frac{5}{4}, 3)$.]

26. Prove that the product of the ordinates of the extremities of a focal chord of a parabola is constant, and deduce that the normals at the extremities of any focal chord are at right angles.

[Let the equation of parabola be $y^2 = 4ax$. Equation of any chord which passes through focus $(a, 0)$ *i. e.* equation of focal chord is $y = m(x - a)$.

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of extremities of focal chord.

Solving $y^2 = 4ax$ and $y = m(x - a)$, the co-ordinates of extremities can be obtained, and eliminating x, between two equations, $my^2 - 4ay - 4a^2m = 0$.

Product of ordinates of extremities of focal chord

$$\equiv y_1 y_2 = -\frac{4a^2 m}{m} = -4a^2 \equiv \text{constant}.$$

Again equations of normals at (x_1, y_1) and (x_2, y_2) are

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \text{ and } y - y_2 = -\frac{y_2}{2a}(x - x_2),$$

'm' of the first line being $m_1 = -\frac{y_1}{2a}$ and that of second being $m_2 = -\frac{y}{2a}$.

$$\therefore \quad m_1 m_2 = +\frac{y_1 y_2}{4a^2} = -\frac{4a^2}{4a^2} = -1.$$

Hence normals at extremities of focal chord are perpendicular.]

27. Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

[Let the equation of parabola be $y^2 = 4ax$ and the co-ordinates of point P on parabola at which ordinate is equal to abscissa is $P(4a, 4a)$.

Equation of normal chord at $(4a, 4a)$ is $y-4a=-\frac{4a}{2a}(x-4a)$ or $y+2x=12a$.

Solving $y^2=4ax$ and $y+2x=12a$, the points at which chords intersect parabola are obtained and co-ordinates of points of intersection are $P(4a, 4a)$ and $Q(9a, -6a)$.

$SP^2=25a^2$; $SQ^2=100a^2$; $PQ^2=125a^2$ where S is focus $(a, 0)$

$\therefore$ $SP^2+SQ^2=PK^2$. Hence the result.

28. Find the equation to the common tangent of the parabolas $y^2=32x$ and $x^2=4y$.

29. Prove that the common tangents of the parabola $y^2=4ax$ and the circle $x^2+y^2-2ax=3a^2$ are both inclined at $30°$ to the x-axis.

30. Show that the sum of the ordinates of the extremities of any chord of a parallel system is constant.

[Consider chords parallel to $y=mx$.

Equation of any chord parallel to $y=mx$ is $y=mx+c$.

Solving $y=mx+c$ and $y^2=4ax$, co-ordinates of extremities are obtained.

Eliminating x between two equations,

$$y^2=4\frac{a}{m}(y-c) \quad \text{or,} \quad my^2-4ay+\frac{4ac}{m}=0.$$

If (x_1, y_1) and (x_2, y_2) are co-ordinates of extremities

$$y_1+y_2=\frac{4a^2}{m^2}=\text{constant}.$$

[Since m for all parallel chords remain same.]

ANSWERS

1. $(2, 6)$. **2.** 18. **3.** $y=2x$. **5.** 8.

6. (i) $25\{(x-5)^2+(y-3)^2\}=(3x-4y+1)^2$.

(ii) $4x^2-4xy+y^2+104x+148y-124=0$.

7. (i) $(-1, 2)$; $(0, 2)$; 4. (ii) $(4, 4\frac{1}{2})$; $(4, 4)$ 2.

8. $y=\pm(x+a)$. **9.** (i) $3x-4y+12=0$. (ii) $y=x+3$.

12. $x=-1$. **14.** $(3, 6)$. **15.** $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$.

16. $x-4y+28=0$; $(28, 14)$. **17.** $y=2x+1$. **18.** $(6, -4\sqrt{3})$.

19. $x(x^2+y^2)+ay^2=0$. **20.** $4x+3y+1=0$. **22.** $4\sqrt{6}$.

23. 80. **24.** $8a\sqrt{2}$. **25.** $(\frac{9}{4}, 3)$. **28.** $2x+y+4=0$.

CHAPTER VII

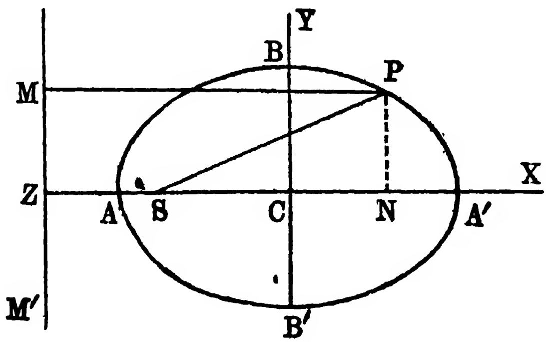
ELLIPSE

7·1. Ellipse.

An *ellipse* is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being less than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (less than unity in this case) is called the *eccentricity* of the ellipse.

7·2. Standard equation of an ellipse.



Let S be the focus, MM' the directrix, and $e(<1)$ the given eccentricity of the ellipse.

Draw SZ perpendicular from S on MM', and let it be divided internally at A and externally at A' in the ratio $e:1$. As $e<1$, $SA' < A'Z$, and accordingly A' is to the right of S as in the figure, on the same side of the directrix MZM' as A, S being between A and A'. Then $SA = e.AZ$ and $SA' = e.A'Z$. Hence, by definition of the ellipse, A and A' are points on the ellipse.

Let C be the middle point of AA'.

Thus, $SA + SA' = e(AZ + A'Z)$ and $SA' - SA = e(A'Z - AZ)$.

Hence, AA' or $2CA = e.2CZ$ and $2CS = e.AA' = e.2CA$.

Let $\quad CA(=CA') = a$.

Then $CZ = \frac{a}{e}$ and $CS = ae$. Let us choose C as origin, and CX along AA' as x-axis, the y-axis CY being $B'CB$ perpendicular to AA' through C.

Now P being any point on the ellipse whose co-ordinates are (x, y), let PN be the perpendicular from P to the x-axis AA', and PM be perpendicular to the directrix MM'. Then $CN = x$, $PM = ZN = ZC + CN = \frac{a}{e} + x$. Also co-ordinates of S are evidently $(-ae, 0)$ [$\because CS = ae$].

Hence, from the property of the ellipse,

$$SP = e.\,PM \quad \text{or} \quad SP^2 = e^2.\,PM^2.$$

$$(x + ae)^2 + y^2 = e^2\left(\frac{a}{e} + x\right)^2$$

or, $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$, ($\because e < 1$ here)

or, writing $a^2(1 - e^2) = b^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \cdots \qquad \cdots \qquad \cdots \quad (1)$$

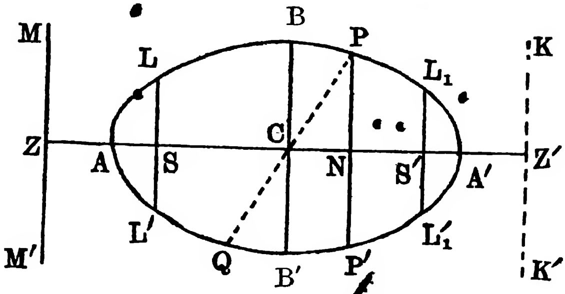
This being the relation satisfied by the co-ordinates of any point on the ellipse, it represents the equation to the ellipse in its standard form.

Here, C, the middle point of AA' (called the *centre*) is the origin, $CA = CA' = \frac{1}{2}AA' = a$, and $b^2 = a^2(1 - e^2)$.

7·3. Shape and elementary properties of the ellipse.

From the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it is apparent that corresponding to any value of x, there are two equal and opposite values of y, namely $\pm \frac{b}{a}\sqrt{a^2 - x^2}$. Hence, on a line perpendicular to AA', corresponding to any point P on one side of it, there is another symmetrical point P' on the other side. Thus

every line perpendicular to AA' is bisected by it, and accordingly *the curve is symmetrical with respect to the x-axis.*



Similarly, for every value of y we get two equal and opposite values of x, and thus *the curve is symmetrical with respect to the y-axis also.* Accordingly if we take points S', Z' on the x-axis, such that $CS'=CS$ and $CZ'=CZ$ on opposite sides of C, and draw $KZ'K'$ parallel to MZM', the ellipse, from symmetry about BCB', can as well be described with S' as focus and KK' as directrix, e being the same. Thus *there is a second focus and a second directrix of the ellipse* symmetrically situated with respect to C.

Again, from the equation to the ellipse, for $y=0$ we get $x=\pm a$. Hence, the ellipse cuts the x-axis at points A' and A given by $x=a$ and $x=-a$ respectively. Similarly, for $x=0$, we get $y=\pm b$, so that the ellipse cuts the y-axis at B and B' given by $y=b$ at B and $y=-b$ at B', so that $CB=CB'=b$ in length.

Moreover, from the equation of the ellipse, if $x>a$ or $<-a$, $\frac{x^2}{a^2}>1$ and so y^2 is negative, and hence y is imaginary. Thus there are no points of the ellipse beyond A' to the right, or beyond A to the left. Similarly, if $y>b$ or $<-b$, x is imaginary, and thus there are no points of the ellipse above B or below B' in the y-direction. Hence, *the ellipse is limited in all directions, and is a closed curve.*

Lastly, from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of the ellipse, if (x_1, y_1) be the co-ordinates of a point P on the ellipse which satisfy the

equation, the co-ordinates $(-x_1, -y_1)$ will also satisfy it, and accordingly the diagonally opposite point Q, where PQ is bisected at C, is also a point on the ellipse. Thus *every chord of the ellipse through C is bisected at C*, and thus the ellipse is symmetrical with respect to the origin C, the mid-point of AA' or BB'. That is why C is called the *centre* of the ellipse.

The length $AA' = 2a$ along the x-axis is called the *major axis* of the ellipse.

The length $BB' = 2b$ along the y-axis is called the *minor axis* of the ellipse.

The points A, A', B, B' are called the *vertices* of the ellipse.

The chord LSL' through the focus S, or $L_1S'L_1$ through the focus S', perpendicular to the major axis AA' (*i.e.*, parallel to the directrix) is called the *latus rectum* of the ellipse, both being of same length by symmetry.

Now ae being the length CS', the x-co-ordinate of the extremity L_1 or L'_1 of the latus rectum is ae. Hence, from the equation to the ellipse, the y-co-ordinate of L_1 or L'_1 is given by $\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$.

Hence, $y = \pm b\sqrt{1-e^2} = \pm a(1-e^2)$.

Thus the *length* $L_1L'_1$, or LL' of the latus rectum

$$= 2a(1-e^2) = 2\frac{b^2}{a}.$$

$$\therefore \quad \text{Semi-latus rectum} = \frac{b^2}{a} = a(1-e^2).$$

Co-ordinates of the extremity L_1, of the latus rectum are $[\,ae, a(1-e^2).\,]$

The eccentricity of the ellipse is given by

$$b^2 = a^2(1-e^2) \quad \text{or} \quad e^2 = \frac{a^2-b^2}{a^2}.$$

Lengths of the focal distances S′P, SP of any point P on the ellipse :

Let the co-ordinates of P be (x_1, y_1). Those of S' being $(ae, 0)$, we get

$$S'P^2 = (x_1 - ae)^2 + y_1{}^2 = (x_1 - ae)^2 + b^2\left(1 - \frac{x_1{}^2}{a^2}\right)$$

[from the equation to the ellipse]

$$= (x_1 - ae)^2 + (1 - e^2)(a^2 - x_1{}^2),$$

[$\because$ $b^2 = a^2(1 - e^2)$]

$$= e^2 x_1{}^2 - 2x_1 ae + a^2 = (a - ex_1)^2.$$

$\therefore$ $\mathbf{S'P = a - ex_1}$, which is the positive value of $S'P$, $\because$ $x_1 < a$, as also $e < 1$.

Similarly, $\mathbf{SP = a + ex_1}$.

Thus, $SP + S'P = 2a =$ length of the major axis. Hence, we get an important property of the ellipse, namely, *the sum of the focal distances of any point on the ellipse is constant and equal to the major axis.*

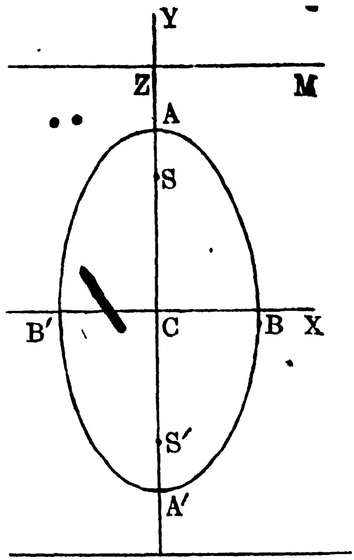
Cor. Focal distance of an extremity of the minor axis is equal to the semi-major axis.

Note 1. As the whole figure is symmetrical with respect to the minor axis BCB', henceforth, for convenience, as a matter of convention, we shall denote the right-hand focus $(ae, 0)$ as S, the right-hand vertex $(a, 0)$ as A, and the left-hand focus $(-ae, 0)$ as S', the left-hand vertex $(-a, 0)$ as A', the right-hand directrix having equation $x = \frac{a}{e}$ being denoted by MZM', and left-hand directrix $x = -\frac{a}{e}$ as $KZ'K'$.

Note 2. The equation $\mathbf{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$, $\mathbf{a > b}$.

If in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the axes of x and y be interchanged, the equation becomes $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. Here, a being greater than b, the major axis having length $2a$ is along the y-axis, and the minor axis of

length $2b$ is along the x-axis. The foci, being on the major axis, *i.e.*, y-axis, will have co-ordinates, $(0, \pm\sqrt{a^2-b^2}.)$ The eccentricity is as before



$e=\sqrt{(a^2-b^2)}/a$. The directrices being parallel to the minor axis *i.e.*, x-axis here, are given by $y=\pm\dfrac{a}{e}$.

7·4. Equation to the tangent at a given point (x_1, y_1) on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Let P be the point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{(i)}$$

and let Q (x_2, y_2) be a neighbouring point on it. The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \text{(ii)}$$

Now since P and Q both lie on the ellipse (i), we have,

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \qquad \cdots \qquad \cdots \text{(iii)}$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1. \qquad \cdots \qquad \cdots \text{(iv)}$$

Hence, subtracting,

$$\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0, \quad \text{or,} \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}.$$

$\therefore$ equation (ii) can be written as

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1) \qquad \cdots \qquad \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the straight line PQ becomes the tangent at P, whose equation [*from (v)*] then becomes

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}(x-x_1),$$

or, $$(y-y_1)\frac{y_1}{b^2}+\frac{x_1}{a^2}(x-x_1)=0,$$

or, $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1. \quad [\text{ by (iii) }]$$

Hence, the equation to the tangent at (x_1, y_1) to the ellipse (i) is

$$\mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.}$$

7·5. Equation to the normal at (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The tangent at (x_1, y_1) to the ellipse is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$,

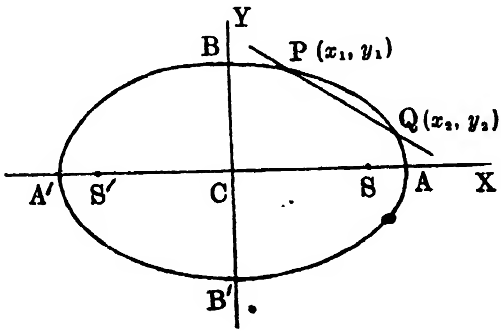
or, $y=-\dfrac{b^2x_1}{a^2y_1}x+\dfrac{b^2}{y_1}$, of which the '$m$' is $-\dfrac{b^2x_1}{a^2y_1}$.

The normal, which is perpendicular to the tangent through (x_1, y_1), has its 'm' $= \frac{a^2 y_1}{b^2 x_1}$, and accordingly its equation is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1),$$

or, $$\frac{x - x_1}{} = \frac{y - y_1}{\frac{y_1}{b^2}}.$$

7·6. Length of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, intercepted by the straight line $y = mx + c$.



At the points of intersection of the line with the ellipse, both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

or, $$x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) + \frac{2mc}{b^2}x + \left(\frac{c^2}{b^2} - 1\right) = 0, \qquad \text{(i)}$$

which being a quadratic equation in x, there are only two values of x and accordingly, only two points of intersection of the given straight line with the ellipse (real, coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then x_1 and x_2 are the roots of (i).

$$\therefore \quad x_1 + x_2 = -\frac{2mc}{b^2}\Big/\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right) = -\frac{2mca^2}{a^2m^2+b^2}$$

$$\text{and } x_1x_2 = \left(\frac{c^2}{b^2}-1\right)\Big/\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right) = \frac{a^2(c^2-b^2)}{a^2m^2+b^2}.$$

$$\therefore \quad (x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$$

$$= \frac{4m^2c^2a^4}{(a^2m^2+b^2)^2} - \frac{4a^2(c^2-b^2)}{a^2m^2+b^2}$$

$$= \frac{4a^2\{m^2c^2a^2-(c^2-b^2)(a^2m^2+b^2)\}}{(a^2m^2+b^2)^2}$$

$$= \frac{4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)^2}.$$

Again P and Q lying on the given line,

$$y_1 = mx_1 + c,\ y_2 = mx_2 + c. \quad \therefore \quad y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$ length of the chord PQ

$$= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(x_1-x_2)^2(1+m^2)}$$

$$= \sqrt{\frac{4a^2b^2(a^2m^2+b^2-c^2)(1+m^2)}{(a^2m^2+b^2)^2}}$$

$$= \frac{\mathbf{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}}{\mathbf{a^2m^2+b^2}}.$$

Cor. Condition of tangency.

The given line will touch the ellipse only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y=mx+c$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $a^2m^2+b^2-c^2=0$, or, $\mathbf{c=\pm\sqrt{a^2m^2+b^2}}$.

7·7. *To show that* $y = mx + \sqrt{a^2m^2+b^2}$ *is a tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *for all values of* m*, and to find the point of contact.*

The tangent at (x_1, y_1) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, or, $y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}$. ... (i)

If the line $y=mx+\sqrt{a^2m^2+b^2}$ ... (ii) be a tangent to the ellipse at (x_1, y_1), the equations (i) and (ii) must be same. Hence, comparing coefficients,

$$-\frac{b^2x_1}{a^2y_1}=m, \frac{b^2}{y_1}=\sqrt{a^2m^2+b^2}.$$

$$\therefore \quad y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}, x_1=-\frac{a^2my_1}{b^2}=-\frac{a^2m}{\sqrt{a^2m^2+b^2}}.$$

The line (ii) therefore will touch the ellipse only if the assumed point (x_1, y_1) is really a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

i.e., if $\left(-\frac{am}{\sqrt{a^2m^2+b^2}}\right)^2+\left(\frac{b}{\sqrt{a^2m^2+b^2}}\right)^2=1$

which is evidently satisfied.

Thus, $y=mx+\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whatever m may be, and the point of contact is given by

$$\mathbf{x_1=-\frac{a^2m}{\sqrt{a^2m^2+b^2}}, y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}}.$$

Similarly, $y=mx-\sqrt{a^2m^2+b^2}$ is also a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for all values of m, the co-ordinates of the point of contact being

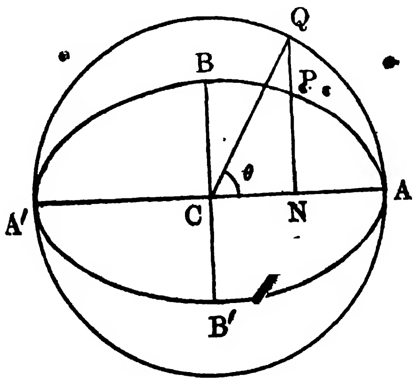
$$\frac{a^2m}{\sqrt{a^2m^2+b^2}}, -\frac{b^2}{\sqrt{a^2m^2+b^2}}.$$

7·8. Auxiliary circle.

The circle on the major axis AA' of an ellipse as diameter is defined as *the auxiliary circle* of the ellipse.

The centre of the circle being the origin C, and its radius being the semi-major axis a, *the equation to the auxiliary circle is*

$$\mathbf{x^2+y^2=a^2.}$$



Let PN be an ordinate of the ellipse, which, when produced, meets the auxiliary circle at Q.

Then x being the abscissa CN of the point P, from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of the ellipse, the ordinate of the ellipse

$$PN=y=\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}=\frac{b}{a}\sqrt{a^2-x^2}$$

Also, from the equation $x^2+y^2=a^2$ of the auxiliary circle, the abscissa $CN=x$ being the same, the ordinate $QN=\sqrt{a^2-x^2}$

Thus, $\frac{PN}{QN}=\frac{b}{a}$.

Hence, *the ratio of any ordinate of the ellipse to the corresponding ordinate of its auxiliary circle is always the same*, and is equal to the ratio of the minor axis to the major axis of the ellipse.

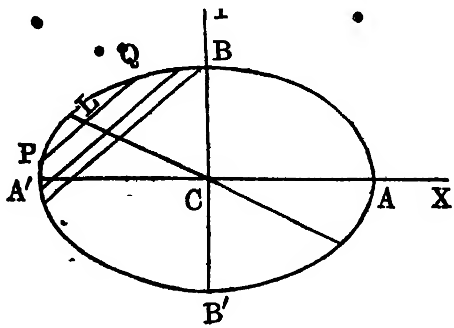
Note. *Co-ordinates of any point on an ellipse expressed in terms of a single variable* ; **eccentric angle** *of a point on the ellipse.*

Let $\angle QCN=\theta$. Then, $\because$ $CQ=a$, clearly $CN=a\cos\theta$, $NQ=a\sin\theta$.

$\therefore$ $NP=\frac{b}{a}\cdot a\sin\theta=b\sin\theta$. Thus, co-ordinates of any point P on the

ellipse can be written as $a\cos\theta$, $b\sin\theta$, in terms of a single variable. Here, θ is called the *eccentric angle* of the point P on the ellipse.

7·9. Locus of the middle points of a *system* of parallel chords ; diameter.



Let PQ, given by the equation $y = mx + c$ ... (i)

be any one of a system of parallel chords of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \ldots \quad \ldots \quad \ldots \text{(ii)}$$

As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

$$\text{or,} \quad (a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0 \quad \ldots \text{(iii)}$$

Thus, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, x_1, x_2 are the roots of (iii), and so $x_1 + x_2 = -\dfrac{2a^2mc}{a^2m^2 + b^2}$. Hence, if (X, Y) be the co-ordinates of the mid-point L of PQ,

$$X = \tfrac{1}{2}(x_1 + x_2) = -\frac{a^2mc}{a^2m^2 + b^2}.$$

Also, $\because$ L is a point on (i), $Y = mX + c$.

$\therefore$ eliminating c,

$$Y = mX - \frac{a^2m^2+b^2}{a^2m}X = -\frac{b^2}{a^2m}X,$$

which is independent of c, and so holds for the middle point of any chord of the parallel system.

Hence, the locus of middle points of a system of parallel chords of the ellipse, parallel to $y=mx$, is

$$y = -\frac{b^2}{a^2m}x$$

which is evidently a straight line passing through the origin, *i.e.*, the centre C of the ellipse. This straight line is called a *diameter* of the ellipse. For different values of m (*i.e.*, for differently directed system of parallel chords) we get different diameters, all passing through the centre.

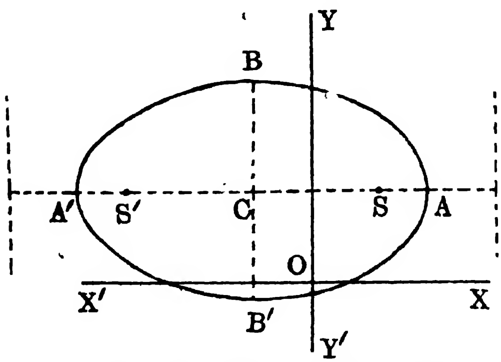
7·10. Illustrative Examples.

Ex. 1. *Show that the equation* $5x^2+9y^2+10x-36y-4=0$ *represents an ellipse, and find its eccentricity, latus rectum, and co-ordinates of the foci. Find also the equations to its directrices.*

The given equation can be written as

$$5(x^2+2x)+9(y^2-4y)=4, \quad \text{or,} \quad 5(x+1)^2+9(y-2)^2=45,$$

i.e., $$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{5}=1.$$



Transferring the origin to the point $(-1, 2)$, the equation reduces to

$$\frac{x^2}{9}+\frac{y^2}{5}=1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

which is the standard form of the equation to an ellipse, with centre as origin.

Hence, the given equation represents an ellipse whose centre (the new origin) has co-ordinates $(-1, 2)$ referred to the original axes of co-ordinates.

Comparing equation (i) with the standard equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ we notice that for (i), } a^2=9,\ b^2=5.$$

Thus, eccentricity of the given ellipse is

$$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{9-5}{9}}=\frac{2}{3}.$$

$$\text{Latus rectum}=2\frac{b^2}{a}=2\cdot\frac{5}{3}=3\tfrac{1}{3}.$$

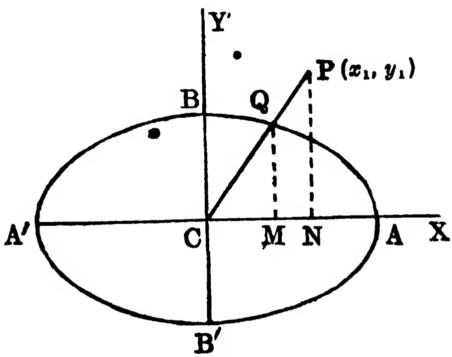
Co ordinates of the foci referred to the centre are

$$(\pm ae, 0),\ i.e.,\ (\pm 3\cdot\tfrac{2}{3}, 0),\ \text{or}\ (\pm 2, 0).$$

Hence, referred to the original axes, the co-ordinates of the foci are $(-1+2, 2+0)$ and $(-1-2, 2+0)$ *i.e.*, $(1, 2)$ and $(-3, 2)$ respectively.

Also, the equations to the directrices referred to centre are $x=\pm\dfrac{a}{e}$ or, $x=\pm\dfrac{3}{\frac{2}{3}}=\pm\dfrac{9}{2}$. Hence, referred to original axes, the equations to the diricrices are $x=\pm\frac{9}{2}-1$ *i.e.*, $x=\frac{7}{2}$ and $x=-\frac{11}{2}$ respectively.

Ex. 2. *Prove that the point* (x_1, y_1) *is inside or outside the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *according as* $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}<1$ *or* >1.



Let P be the point where co-ordinates are (x_1, y_1) and let CP, the line joining the centre (origin) to P intersect the ellipse at Q. Then, if $\dfrac{CP}{CQ}=\lambda$, clearly P is outside the ellipse if $\lambda>1$ and inside if $\lambda<1$.

Now, PN and QM being perpendiculars on the x-axis CAX, clearly $x_1 = CN$, $y_1 = NP$, and $\frac{CM}{CN} = \frac{MQ}{NP} = \frac{CQ}{CP} = \frac{1}{\lambda}$.

$\therefore$ the co-ordinates of Q, namely CM and MQ are respectively $\frac{x_1}{\lambda}, \frac{y_1}{\lambda}$.

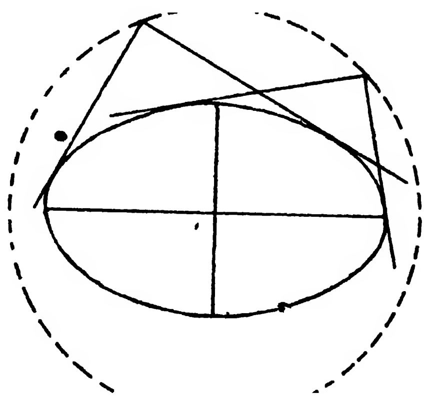
As Q lies on the ellipse, its co-ordinates will satisfy the equation to the ellipse, and hence

$$\frac{x_1^2}{\lambda^2 a^2} + \frac{y_1^2}{\lambda^2 b^2} = 1, \text{ or, } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \lambda^2.$$

Hence, the point P is outside or inside the ellipse according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$ or < 1.

Ex. 3. *Prove that the locus of point of intersection of any two perpendicular tangents to an ellipse is a circle.*

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \cdots$ (i) be the equation to the ellipse.



Any tangent to (i) is $y = mx + \sqrt{a^2m^2 + b^2}$ (ii). For the perpendicular tangent, replacing m by $-\frac{1}{m}$, the equation is $y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} + b^2}$,

$my = -x + \sqrt{a^2 + b^2m^2}$ ... (iii)

At the point of intersection of (ii) and (iii) both the equations are satisfied. Hence, if we eliminate m between these two equations, the rel

tion obtained will be satisfied at the points of intersection of every such pair of perpendicular tangents, and will thus represent the equation to the desired locus.

Now, from (ii) and (iii),

$$y-mx=\sqrt{a^2m^2+b^2} \text{ and } my+x=\sqrt{a^2+b^2m^2}.$$

Squaring and adding,

$$(x^2+y^2)(1+m^2)=(a^2+b^2)(1+m^2).$$

$$\therefore \quad x^2+y^2=a^2+b^2$$

which evidently represents a circle with its centre at the centre of the ellipse.

Note. The circle is known as the **director circle** of the ellipse.

Ex. 4. *Find the length of the chord of the ellipse* $\frac{x^2}{25}+\frac{y^2}{16}=1$, *whose middle point is* $(\frac{1}{2}, \frac{2}{5})$.

Let the equation to the chord PQ, whose middle point is $(\frac{1}{2}, \frac{2}{5})$ be

$$y-\tfrac{2}{5}=m(x-\tfrac{1}{2}), \text{ or } y=mx+\frac{4-5m}{10}. \quad \dots \quad \dots \quad \text{(i)}$$

The abscissæ of its points of intersection P and Q with the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \quad \dots \quad \dots \quad \dots \quad \dots \quad \text{(ii)}$$

are given by [eliminating y between (i) and (ii)]

$$\frac{x^2}{25}+\frac{1}{16}\left\{mx+\frac{4-5m}{10}\right\}^2=1,$$

or,

$$(16+25m^2)x^2+5m(4-5m)x+\frac{(4-5m)^2-1600}{4}=0. \quad \dots \text{(iii)}$$

Now, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, then x_1, x_2 are the roots of (iii).

Hence, $$x_1+x_2=\frac{5m(5m-4)}{16+25m^2} \quad \dots \quad \dots \quad \dots \text{(iv)}$$

and $$x_1x_2=\frac{(4-5m)^2-1600}{4(16+25m^2)}. \quad \dots \quad \dots \quad \dots \text{(v)}$$

But the abscissa of the middle point of PQ is given to be $\frac{1}{2}$.
$\therefore \quad \frac{1}{2}(x_1+x_2)=\frac{1}{2}$, or, $x_1+x_2=1$.

$\therefore$ from (iv), $16+25m^2=5m(5m-4)$. $\therefore$ $m=-\frac{4}{5}$.

$\therefore$ (v) gives $x_1x_2=\dfrac{64-1600}{4.32}=-12.$

$\therefore$ $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=1+48=49. \quad \dots \quad \dots \text{(vi)}$

As P and Q both lie on (i),

$$y_1 - \tfrac{3}{5} = m(x_1 - \tfrac{1}{2}) \quad y_2 - \tfrac{3}{5} = m(x_2 - \tfrac{1}{2}).$$

$$\therefore \quad y_1 - y_2 = m(x_1 - x_2) = -\tfrac{4}{5}(x_1 - x_2).$$

$$\therefore \quad PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + \tfrac{16}{25})}$$

$$= \sqrt{49 \times \tfrac{41}{25}} = \tfrac{7}{5}\sqrt{41}.$$

Ex. 5. *Prove that in the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *if the line* $y = m'x$ *bisects all chords parallel to* $y = mx$, *then* $y = mx$ *also bisects all chords parallel to* $y = m'x$.

As in § 7·9, we see that the bisector of all chords parallel to $y = mx$ is the diameter $y = -\frac{b^2}{a^2 m}x$. Hence, if this diameter is given to be $y = m'x$, we must have $m' = -\frac{b^2}{a^2 m}$ or $mm' = -\frac{b^2}{a^2}$ ... (i), which is the condition that $y = m'x$ may bisect all chords parallel to $y = mx$.

Similarly, the condition that $y = mx$ may bisect all chords of the ellipse parallel to $y = m'x$ is $m'm = -\frac{b^2}{a^2}$, which is identical with (i).

Hence, if $y = m'x$ bisects all chords parallel to $y = mx$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $y = mx$ will also bisect all chords parallel to $y = m'x$, the common condition being $mm' = -\frac{b^2}{a^2}$.

Hence, if a diameter of an ellipse bisects all chords parallel to another diameter, the latter will also bisect all chords parallel to the former.

Two such diameters are referred to as **conjugate diameters** of the ellipse.

Examples VII

1. (i) Find out the eccentricity, and the co-ordinates of the foci of the ellipse $9x^2 + 25y^2 = 225$. [*H. S. 1960*]

(ii) Find the co-ordinates of the foci of the ellipse $9x^2 + 5y^2 = 45$.

2. An ellipse has its major axis along the x-axis and minor axis along the y-axis. Its eccentricity is $\frac{1}{2}$ and the distance between the foci is 4. Find its equation and show that the ellipse passes through the point (2, 3). [*H. S. 1961, Compartmental*]

[Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Distance between the foci $=2ae=4$ or $2.a.\frac{1}{2}=4$ [since $e=\frac{1}{2}$] $\therefore$ $a=4.$

Again $b^2=a^2(1-e^2)=16\,(1-\frac{1}{4})=12.$

Hence the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{12}=1$ which is satisfied by $(2, 3)$.]

3. (i) Find the equation to the ellipse whose centre is the origin, whose axes are the axes of co-ordinates, and which passes through the points $(-3, \frac{16}{5})$ and $(0, -4)$.

Find also the co-ordinates of its foci.

[Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Since it passes through $(-3, \frac{16}{5})$ and $(0, -4)$,

$$\frac{9}{a^2}+\frac{256}{25b^2}=1 \quad \text{and} \quad \frac{16}{b^2}=1.$$

Solving, $a^2=25$ and $b^2=16.$

Hence the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1.$

Again $16=25(1-e^2)$ or $e=\pm\frac{3}{5}.$

$\therefore$ co-ordinates of foci $(\pm ae, 0)$ *i.e.*, $(\pm 5.\frac{3}{5}, 0)$ *i.e.*, $(\pm 3, 0)$.]

(ii) An ellipse having centre as origin and axes along the co-ordinate axes, passes through the points $(\frac{3}{2}, -3)$ and $(-\sqrt{6}, 2)$. Find the equations to its directrices.

4. Find the equation to the ellipse having centre as origin, and axes along the axes of co-ordinates, whose latus rectum is 6 and eccentricity $\frac{1}{2}$. Write down the co-ordinates of the extremities of its minor axis.

5. (i) The latus rectum of an ellipse is half its major axis ; find its eccentricity.

(ii) The distance between the focus and directrix of an ellipse is 16 inches and its eccentricity is $\frac{3}{5}$. Obtain the lengths of its principal axes.

6. Find the equation to the ellipse whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$, and directrix is $x-y+3=0$.

7. Find the latus rectum, eccentricity and co-ordinates of the centre and foci of the ellipse :

(i) $3x^2+4y^2+6x-8y=5$.

(ii) $9x^2+5y^2-30y=0$.

8. Is the point (i) $(2, -1\frac{1}{2})$, (ii) $(2, -1)$, inside or outside the ellipse $4x^2+9y^2=36$?

9. Find the equation to the tangent of the ellipse $9x^2+16y^2=144$ having equal positive intercepts on the axes.

[*H. S. 1961*]

10. Find the distance from the origin of the point where the tangent at the extremity of a latus rectum of the ellipse $9x^2+25y^2=225$ intersects the major axis. [*H. S. 1960*]

11. Show that $x-3y=13$ touches the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1. \qquad \text{[*H. S. 1960, Compartmental*]}$$

What are the co-ordinates of the point of contact ?

[Since $\sqrt{25(\frac{1}{3})^2+16}=\frac{13}{3}$ which is equal to the numerical value of the constant of line $y=\frac{x}{3}-\frac{13}{3}$. [Cor. § 7·6], hence the line touches the ellipse.

Let (x_1, y_1) be the point of contact.

The equation of tangent at (x_1, y_1)

$$\frac{xx_1}{25}+\frac{yy_1}{16}=1 \text{ must be identical with } x-3y=13.$$

$$\frac{x_1/25}{1}=\frac{y_1/16}{-3}=\frac{1}{13}. \quad \therefore \text{ co-ordinates of point of contact } \left(\frac{25}{13}, -\frac{48}{13}\right).]$$

12. Find the equations to the tangents to the ellipse $9x^2+16y^2=36$ which are parallel to $3x-3y+7=0$, and find out the points of contact.

13. If a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepts lengths α and β along the axes, prove that $a^2/\alpha^2+b^2/\beta^2=1$.

[Equation of tangent is $\dfrac{x}{\alpha}+\dfrac{y}{\beta}=1$.

$\therefore$ since tangent to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. $\therefore$ $\beta=\sqrt{a^2\left(-\dfrac{\beta}{\alpha}\right)^2+b^2}$

or, $\dfrac{a^2}{\alpha^2}+\dfrac{b^2}{\beta^2}=1$.]

14. Prove that the product of the perpendiculars from the foci on any tangent to an ellipse is constant, and is equal to the square on the semi-minor axis.

15. The straight line $3x-5y+25=0$ touches an ellipse whose principal axes are along the axes of co-ordinates, and whose eccentricity is given to be $\frac{3}{5}$. Find the distance between the foci of the ellipse.

[Let the equation of ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$\therefore$ $b^2=a^2(1-e^2)=a^2(1-\frac{9}{25})=\frac{16}{25}a^2$. $\quad\cdots\quad\cdots\quad(1)$

Since $3x-5y+25=0$ is a tangent to the ellipse,

$\therefore$ $5=\sqrt{a^2(\frac{3}{5})^2+b^2}$ or, $25=\frac{9}{25}a^2+b^2$. $\quad\cdots\quad\cdots\quad(2)$

Solving (1) and (2), $a=5$.

Co-ordinates of foci $(\pm ae, 0)$ *i.e.*, $(\pm 3, 0)$.

Distance between foci $=2ae=6$.]

16. Find out the equation to the normal to the ellipse $2x^2+7y^2=71$ at $(2, -3)$ and determine the distance of the point where it intersects the major axis, from the foot of the ordinate.

[Equation of ellipse is $\dfrac{x^2}{\frac{71}{2}}+\dfrac{y^2}{\frac{71}{7}}=1$.

As in § 7·5, equation of normal is $\dfrac{x-2}{\dfrac{2}{\frac{71}{2}}}=\dfrac{y+3}{\dfrac{-3}{\frac{71}{7}}}$

or $\quad 21x+4y=30$.

When it intersects major axis, *i.e.*, x-axis, $y=0$.

$\therefore$ $x=\frac{30}{21}$ at $(\frac{30}{21}, 0)$.

Distance of the point $(\frac{30}{21}, 0)$ from the foot of the ordinate $=\frac{30}{21}-2=-\frac{4}{7}$.]

17. Write down the equation to the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at an extremity of the latus rectum, and show that

if it passes through an extremity of the minor axis, the eccentricity of the ellipse is given by $e^2 = \frac{1}{2}(\sqrt{5} - 1)$.

18. If the normal to the ellipse $x^2 + 3y^2 = 12$ at a point be inclined at $60°$ to the major axis, show that the line joining the centre to the point is inclined at $30°$ to the same axis.

19. Obtain the equation to the chord of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which is bisected at the point $(2, -1)$.

20. Find the length of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ intercepted by the line $x + y = 3$. What are the co-ordinates of its middle point ?

21. Find the equation to the diameter of the ellipse $6x^2 + 9y^2 = 1$ bisecting all chords parallel to $y = x$.

[As in § 7·9, equation of diameter is $y = -\frac{b^2}{a^2 m}x$.

Hence $b^2 = \frac{1}{9}$, $a^2 = \frac{1}{6}$ and $m = 1$.

$\therefore$ $y = -\frac{\frac{1}{9}}{\frac{1}{6} \cdot 1} \cdot x$, or $3y + 2x = 0$.]

22. Show that the straight lines $3y = 4x$ and $x + 3y = 0$ each bisects all chords of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ parallel to the other.

ANSWERS

1. (i) $\frac{4}{5}$; $(\pm 4, 0)$. (ii) $(0, \pm 2)$. **2.** $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

3. (i) $\frac{x^2}{25} + \frac{y^2}{16} = 1$; $(\pm 3, 0)$. (ii) $y = \pm 4\sqrt{3}$. **4.** $\frac{x^2}{16} + \frac{x^2}{12} = 1$; $(0, \pm 2\sqrt{3})$.

5. (i) $\frac{1}{\sqrt{2}}$. (ii) 30 inches, 24 inches.

6. $8\{(x+1)^2 + (y-1)^2\} = (x - y + 3)^2$, or, $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$.

7. (i) 3 ; $\frac{1}{2}$; $(-1, 1)$; $(0, 1)$ and $(-2, 1)$. (ii) $3\frac{1}{3}$; $\frac{2}{3}$; $(0, 3)$; $(0, 1)$ and $(0, 5)$.

8. (i) Outside. (ii) Inside. **9.** $x + y = 5$, **10.** $6\frac{1}{4}$. **11.** $(\frac{16}{7}, -\frac{48}{15})$.

12. $2x - 2y = \pm 5$; $(\frac{8}{5}, -\frac{9}{10})$ and $(-\frac{8}{5}, \frac{9}{10})$. **15.** 6. **16.** $21x + 4y = 30$; $-\frac{4}{7}$.

17. $x = e(y + ae^2)$. **19.** $8x - 9y = 25$. **20.** $7\frac{33}{41}$; $(\frac{75}{41}, \frac{48}{41})$. **21.** $2x + 3y = 0$.

CHAPTER VIII

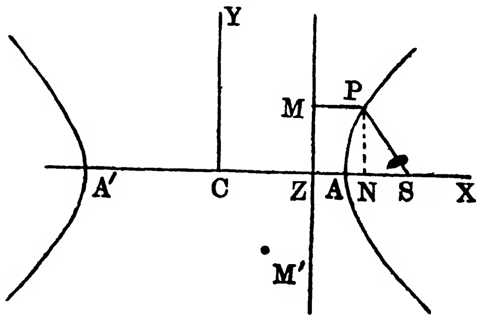
HYPERBOLA

8·1. Hyperbola.

A *hyperbola* is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being greater than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (which is greater than unity in this case) is called the *eccentricity* of the hyperbola.

8·2. Standard equation of a hyperbola.



Let S be the focus, MM' the directrix, and $e(>1)$ the given eccentricity of the hyperbola.

Draw SZ perpendicular from S on MM', and let it be divided internally at A and externally at A' in the ratio $e:1$. As $e>1$, $SA' > A'Z$, and accordingly A' is to the left of S as in the figure, on the side of the directrix MZM' opposite to A, S being not between A and A'.

Then, $SA=e.AZ$ and $SA'=e.A'Z$.

Hence, by definition of the hyperbola, A and A' are points on the hyperbola.

Let C be the middle point of AA'.

Thus, $SA + SA' = e(AZ + A'Z)$

and $SA' - SA = e(A'Z - AZ)$,

or, $2CS = e.AA' = e.2CA$ and AA' or $2CA = e.2CZ$.

Let $CA = CA' = a$. Thus, $CS = ae$, and $CZ = \dfrac{a}{e}$.

Let us choose C as origin, and CX along $A'A$ as x-axis, the y-axis CY being parallel to $M'M$ *i.e.*, perpendicular to $A'A$ through C.

Now, P being any point on the hyperbola, whose co-ordinates are (x, y), let PN be the perpendicular from P to the x-axis, and PM be perpendicular to the directrix MM'. Then, $CN = x$, $PM = ZN = CN - CZ = x - \dfrac{a}{e}$. Also co-ordinates of S are evidently $(ae, 0)$. [$\because$ $CS = ae$].

Hence, from the property of the hyperbola,

$$SP = e.PM, \quad \text{or,} \quad SP^2 = e^2 PM^2.$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

or, $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$, ($\because$ $e > 1$ here)

or, writing $a^2(e^2 - 1) = b^2$,

$$\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.} \qquad \ldots \qquad \ldots \quad \text{(i)}$$

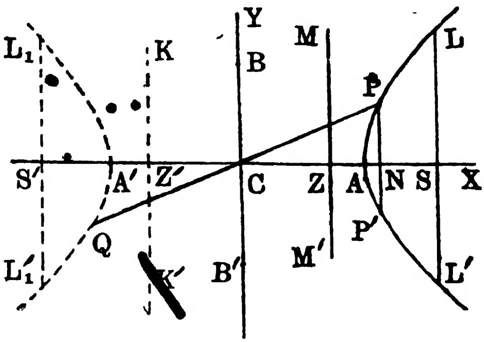
This being the relation satisfied by the co-ordinates of any point on the hyperbola, it represents the equation to the hyperbola in its standard form.

Here C, the middle point of AA' (called the *centre*) is the origin, $CA = CA' = a$, and $b^2 = a^2(e^2 - 1)$.

8·3. Shape and elementary properties of the hyperbola.

From the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ of the hyperbola, the following points may be noted.

If $y=0$, $x=\pm a$, so that the hyperbola cuts the x-axis at points A and A' given by $x=a$ and $x=-a$ respectively.



If $x=0$, y^2 is negative and so y is imaginary. Accordingly the curve does not cut the y-axis at all.

For values of $x < a$ or $> -a$ (*i.e.*, within AA'), $\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$ is negative, and so y is imaginary. Hence, there is no portion of the hyperbola within the range AA'.

For values of $x > a$ or $< -a$, $\frac{x^2}{a^2} > 1$ and so $\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$ is positive. Hence, y has two equal and opposite values. Thus from A to the right, or from A' to the left, the curve extends, being symmetrical with respect to the x-axis, y having greater and greater magnitudes as the magnitude of x becomes greater and greater.

Again, for any values of y, $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}$ is positive, and so x has two equal and opposite values. Hence, the curve is symmetrical with respect to the y-axis.

Thus, *the hyperbola consists of two detached portions* as shown in the figure, extending from A towards the right and from A' toward the left, *being symmetrical about both the x-axis and the y-axis.*

From symmetry about the y-axis CY, we see that if we take points S' and Z' on the x-axis such that $CS'=CS$ and $CZ'=CZ$

on opposite sides of C, and draw $KZ'K'$ parallel to MZM', the curve might be drawn equally well with S' as focus and $KZ'K'$ as directrix, e being the same as before. Hence, *a hyperbola has a second focus and a second directrix* symmetrically situated with respect to C.

Lastly, if (x_1, y_1) be the co-ordinates of a point P on the hyperbola so that they satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the co-ordinates $(-x_1, -y_1)$ will also satisfy it, and accordingly the diagonally opposite point Q, where PQ is bisected at C, is also a point on the hyperbola. Thus, *every chord of the hyperbola through C is bisected at C*, and so the hyperbola is symmetrical with respect to the origin C, the mid-point of AA'. This is why C is called the *centre* of the hyperbola.

The x-axis is here referred to as the *transverse axis*, and the length $AA' = 2a$ is called the length of the transverse axis. The y-axis here is referred to as the *conjugate axis*, and a length $BB' = 2b$ (where $CB = CB' = b$) along this axis is referred to as the *length* of the conjugate axis.

The chord LSL' through the focus S (or $L_1S'L'_1$ through the focus S') perpendicular to the transverse axis (*i.e.*, parallel to the directrix) is called the *latus rectum* of the hyperbola.

Now, ae being the length CS, the x-co-ordinate of the extremity L of the latus rectum is ae. Hence, from the equation to the hyperbola, the y-co-ordinate of L is given by

$$\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Hence, $\quad y = \pm b\sqrt{e^2 - 1} = \pm a(e^2 - 1).$

Thus, *the length LL' of the latus rectum*

$$= 2a(e^2 - 1) = 2\frac{b^2}{a}.$$

$\therefore$ semi-latus rectum $= \frac{b^2}{a} = a(e^2 - 1)$.

Co-ordinates of the extremity L of the latus rectum are

$[ae, a(e^2 - 1).]$

The eccentricity of the hyperbola is given by

$$b^2 = a^2(e^2-1), \text{ or, } e^2 = \frac{a^2+b^2}{a^2}.$$

Note 1. If $a=b$, the hyperbola is said to be a **rectangular or equilateral hyperbola.** For a rectangular hyperbola the eccentricity

$$= \sqrt{2}.$$

Note 2. Lengths of the focal distances SP, S′P of any point P on the hyperbola.

Let the co-ordinates of P be (x_1, y_1). Those of S being $(ae, 0)$ we get

$$SP^2 = (x_1-ae)^2 + y_1^2 = (x_1-ae)^2 + b^2\left(\frac{x_1^2}{a^2}-1\right)$$

[*from the equation to the hyperbola*]

$$= (x_1-ae)^2 + (e^2-1)(x_1^2-a^2) \qquad [\because b^2 = a^2(e^2-1)]$$

$$= e^2x_1^2 - 2x_1ae + a^2 = (ex_1-a)^2.$$

$\therefore$ $\mathbf{SP = ex_1 - a}$, which is the positive value of SP,

$\because$ $x_1 > a$ and $e > 1$ here.

Similarly, $\mathbf{S'P = ex_1 + a}$.

Thus, $S'P - SP = 2a =$ length of the transverse axis.

Hence, we get the important property of the hyperbola, namely, *the difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis.*

8·4. Equation to the tangent at a given point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let P be the point (x_1, y_1) on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots \qquad \ldots \qquad \ldots \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it. The equation to the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \ldots \qquad \ldots \text{(ii)}$$

Now, since P and Q both lie on the hyperbola (i), we have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \ldots \text{(iii)} \qquad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1 \qquad \ldots \text{(iv)}$$

Hence, subtracting,

$$\frac{x_2^2 - x_1^2}{a^2} - \frac{y_2^2 - y_1^2}{b^2} = 0, \text{ or, } \frac{y_2 - y_1}{x_2 - x_1} = \frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1}.$$

$\therefore$ equation (ii) can be written as

$$y - y_1 = \frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1} (x - x_1) \qquad \dots \text{(v)}$$

Now, make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position the straight line PQ becomes the tangent at P whose equation [*from (v)*] then becomes

$$y - y_1 = \frac{b^2}{a^2} \cdot \frac{x_1}{y_1} (x - x_1),$$

or, $$\frac{y_1}{b^2} (y - y_1) = \frac{x_1}{a^2} (x - x_1),$$

or, $$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1. \qquad [\textit{ by (iii)}]$$

Hence, the equation to the tangent at (x_1, y_1) to the hyperbola (i) is

$$\mathbf{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.}$$

8·5. Equation to the normal at $(\mathbf{x_1, y_1})$ to the hyperbola $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$

The tangent at (x_1, y_1) to the hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

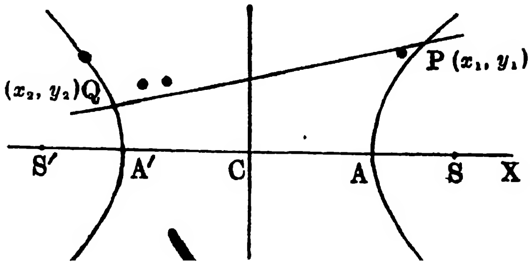
or, $y = \frac{b^2 x_1}{a^2 y_1} x - \frac{b^2}{y_1}$, of which the '$m$' is $\frac{b^2 x_1}{a^2 y_1}$.

The normal which is perpendicular to the tangent through (x_1, y_1), has its 'm' $= -\frac{a^2 y_1}{b^2 x_1}$, and accordingly its equation is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1} (x - x_1),$$

or, $$\mathbf{\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}.}$$

8·6. Length of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, intercepted by the straight line $y=mx+c$.



At the points of intersection of the line with the hyperbola, both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1,$$

or, $$(a^2m^2-b^2)x^2+2mca^2x+a^2(b^2+c^2)=0 \qquad \text{(i)}$$

which being a quadratic in x, there are only two values of x and accordingly only two points of intersection of the given straight line with the hyperbola (real, coincident or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then, x_1 and x_2 are the roots of (i).

$$\therefore \quad x_1+x_2=-\frac{2mca^2}{a^2m^2-b^2} \text{ and } x_1x_2=\frac{a^2(b^2+c^2)}{a^2m^2-b^2}.$$

$$\therefore \quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{4m^2c^2a^4}{(a^2m^2-b^2)^2}-\frac{4a^2(b^2+c^2)}{a^2m^2-b^2}$$

$$=\frac{4a^2\{m^2c^2a^2-(b^2+c^2)(a^2m^2-b^2)\}}{(a^2m^2-b^2)^2}$$

$$=\frac{4a^2b^2(c^2-a^2m^2+b^2)}{(a^2m^2-b^2)^2}$$

Again, P and Q lying on the given line $y = mx + c$,

$$y_1 = mx_1 + c,\ y_2 = mx_2 + c. \quad \therefore\ y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$ length of the chord PQ

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + m^2)}$$

$$= \sqrt{\frac{4a^2b^2(c^2 - a^2m^2 + b^2)(1 + m^2)}{(a^2m^2 - b^2)^2}}$$

$$= \frac{\mathbf{2ab\sqrt{1+m^2}\sqrt{c^2 - a^2m^2 + b^2}}}{\mathbf{a^2m^2 - b^2}}.$$

Cor. Condition of tangency.

The given line will touch the hyperbola only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y = mx + c$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$c^2 - a^2m^2 + b^2 = 0, \text{ or, } \mathbf{c = \pm\sqrt{a^2m^2 - b^2}}.$$

8·7. *To show that $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of m, and to find the point of contact.*

The tangent at (x_1, y_1) of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2}$

$$- \frac{yy_1}{b^2} = 1, \quad \text{or, } y = \frac{b^2x_1}{a^2y_1}x - \frac{b^2}{y_1}. \quad \cdots \quad \text{(i)}$$

If the line $y = mx + \sqrt{a^2m^2 - b^2}$ $\cdots$ (ii) be a tangent to the hyperbola at (x_1, y_1), the equations (i) and (ii) must be identical. Hence, comparing coefficients,

$$\frac{b^2x_1}{a^2y_1} = m, \ -\frac{b^2}{y_1} = \sqrt{a^2m^2 - b^2}\ ;$$

$$y_1 = -\frac{b^2}{\sqrt{a^2m^2 - b^2}},\ x_1 = \frac{ma^2y_1}{b^2} = -\frac{ma^2}{\sqrt{a^2m^2 - b^2}}.$$

The line (ii) therefore will touch the hyperbola only if the assumed point (x_1, y_1) is really a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *i.e.*, if

$$\left(-\frac{am}{\sqrt{a^2m^2-b^2}}\right)^2 - \left(\frac{-b}{\sqrt{a^2m^2-b^2}}\right)^2 = 1$$

which is evidently satisfied.

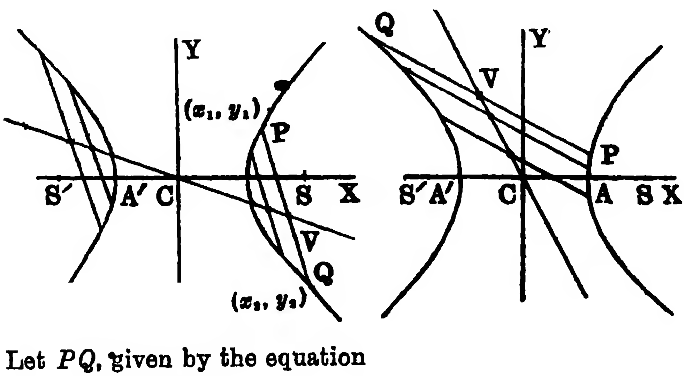
Thus, $y = mx + \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whatever m may be, and the point of contact is given by

$$x_1 = -\frac{a^2m}{\sqrt{a^2m^2-b^2}} \qquad y_1 = -\frac{b^2}{\sqrt{a^2m^2-b^2}}$$

Similarly, $y = mx - \sqrt{a^2m^2-b^2}$ is also a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, for all values of m, the co-ordinates of the point of contact being

$$\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{b^2}{\sqrt{a^2m^2-b^2}}\right).$$

8·8. Locus of the middle points of a system of parallel chords ; diameter.



Let PQ, given by the equation

$$y = mx + c \qquad \ldots$$

be any one of a system of parallel chords of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \cdots \text{ (ii)}$$

As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or, $$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(b^2 + c^2) = 0. \qquad \cdots \text{ (iii)}$$

Thus, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, x_1, x_2 are the roots of (iii), and so.

$$x_1 + x_2 = -\frac{2a^2mc}{a^2m^2 - b^2}.$$

Hence, if (X, Y) be the co-ordinates of the mid-point V of PQ,

$$X = \tfrac{1}{2}(x_1 + x_2) = -\frac{a^2mc}{a^2m^2 - b^2}.$$

Also, $\because$ V is a point on (i), $Y = mX + c$.

$\therefore$ eliminating c,

$$X = \frac{-a^2m(Y - mX)}{a^2m^2 - b^2}, \text{ or, } -b^2X = -a^2mY,$$

or, $$Y = \frac{b^2}{a^2m}X,$$ which being independent of c holds for the middle point of any chord of the parallel system. Hence, the locus of the middle points of a system of parallel chords of the hyperbola, parallel to $y = mx$, is

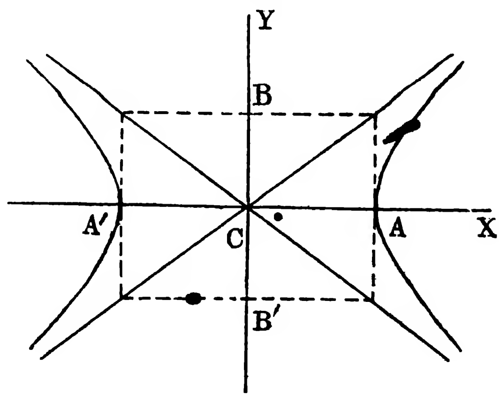
$$\mathbf{y} = \frac{\mathbf{b^2}}{\mathbf{a^2m}}\mathbf{x}$$

which is evidently a straight line passing through the origin *i.e.*, the centre C of the hyperbola. This straight line is called a *diameter* of the hyperbola. For different values of m (*i.e.*, for differently directed system of parallel chords) we get different diameters, all passing through the centre.

8·9. Asymptotes of a hyperbola.

We have noticed in § 8·7 that the line given by $y = mx + \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and the co-ordinates of the point of contact are $\left(-\frac{a^2m}{\sqrt{a^2m^2-b^2}}, -\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$. Now, if m be so chosen that $a^2m^2 - b^2 = 0$, or, $m = \pm\frac{b}{a}$, the co-ordinates of the point of contanct both tend to infinity.

Hence, the straight lines $y = \pm\frac{b}{a}x$ are both tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the points of contact tend to infinity. These lines are defined as *asymptotes* to the hyperbola.



They are inclined to the transverse axis at an angle θ, where $\tan\theta = \pm\frac{b}{a}$. Hence, with sides equal and parallel to the transverse axis $2a$ and conjugate axis $2b$ of the hyperbola, if we construct a rectangle with centre at the origin, the diagonals will be the asymptotes, which will continually approach the hyperbola, and will ultimately touch it at infinite distance.

In the particular case when $b=a$, the asymptotes are inclined to the x-axis at angles $\pm 45°$, and so they are mutually perpendicular. The hyperbola in this case, when its transverse and conjugate axes are equal in length, is defined to be a *rectangular or equilateral hyperbola*, having its asymptotes mutually perpendicular.

8·10. Illustrative Examples.

Ex. 1. *The co-ordinates of the foci of a hyperbola are* $(-5, 3)$ *and* $(7, 3)$, *and its eccentricity is* $\frac{3}{2}$. *Find the equation and determine the length of its latus rectum.*

Let $S\,(7, 3)$ and $S'\,(-5, 3)$ be the given foci. The eccentricity $e=\frac{3}{2}$. If $2a$ be the length of the transverse axis, then $SS'=2ae$,

$$\text{or,}\quad 12=2a\times\tfrac{3}{2}. \qquad \therefore\ a=4.$$

Also, $2b$ being the conjugate axis,

$$b^2=a^2(e^2-1)=16\,(\tfrac{9}{4}-1)=20.$$

Hence, the length of the latus rectum

$$=2\frac{b^2}{a}=2.\frac{10}{4}=10.$$

Again, the middle point C of SS' is clearly the centre, and its co-ordinates are

$$\tfrac{1}{2}(7-5) \text{ and } \tfrac{1}{2}(3+3),\ i.e.,\ (1, 3).$$

Also, the transverse axis, which is along the line SS' has equation

$$(y-3)(7+5)=(x-7)(3-3)=0, \quad i.e., \quad y=3$$

and hence it is parallel to the x-axis.

Now referred to the centre C as origin, and transverse axis as x-axis, the equation to the hyperbola (whose $a^2=16$ and $b^2=20$) is evidently

$$\frac{x^2}{16}-\frac{y^2}{20}=1.$$

Hence, referred to given axes [with reference to which C has co-ordinates $(1, 3)$, and to which the transverse and conjugate axes of the hyperbola are parallel] the required equation to the hyperbola is evidently

$$\frac{(x-1)^2}{16}-\frac{(y-3)^2}{20}=1. \qquad \ldots \qquad \ldots \quad \text{(i)}$$

Alternatively

Since the difference of the focal distances of any point on the hyperbola is equal to its transverse axis, which is $2a=8$ here, if (x, y) be the co-ordinates of any point on the hyperbola,

$$\sqrt{(x+5)^2+(y-3)^2} \sim \sqrt{(x-7)^2+(y-3)^2}$$

or, $$\sqrt{(x+5)^2+(y-3)^2}=\sqrt{(x-7)^2+(y-3)^2} \pm 8.$$

Hence, squaring and transposing,

$$24x-88= \pm 16 \sqrt{(x-7)^2+(y-3)^2},$$

or, $$(3x-11)^2=4\{(x-7)^2+(y-3)^2\},$$

or, $$5x^2-4y^2-10x+24y-111=0$$

which is the required equation to the hyperbola, and is the same as equation (i) already obtained above.

Ex. 2. *Prove that the tangent to the hyperbola* $x^2-3y^2=12$ *at the point* $(-6, 2\sqrt{2})$ *bisects the angle between the focal distances of the point.*

The given equation to the hyperbola can be written in the form

$$\frac{x^2}{12}-\frac{y^2}{4}=1 \qquad \cdots \qquad \cdots \qquad \cdots \quad \text{(i)}$$

and hence the co-ordinates of its foci S and S' are easily seen to be

$$(\pm\sqrt{12+4}, 0) \text{ i.e., } (\pm 4, 0).$$

Now P being the point $(-6, 2\sqrt{2})$ on the hyperbola, the equations to the focal distances SP and $S'P$ are

$$y=\frac{2\sqrt{2}}{-6-4}(x-4), \text{ i.e., } x\sqrt{2}+5y-4\sqrt{2}=0 \qquad \cdots \quad \text{(ii)}$$

and $$y=\frac{2\sqrt{2}}{-6+4}(x+4), \text{ i.e., } x\sqrt{2}+y+4\sqrt{2}=0 \qquad \cdots \quad \text{(iii)}$$

respectively.

The equation to the bisector of the angle SPS', *i.e.*, between (ii) and (iii), in which the origin lies, is

$$\frac{x\sqrt{2}+5y-4\sqrt{2}}{-\sqrt{2+25}}=\frac{x\sqrt{2}+y+4\sqrt{2}}{\sqrt{2+1}}$$

or, $$x\sqrt{2}+5y-4\sqrt{2}+3(x\sqrt{2}+y+4\sqrt{2})=0,$$

or, $$x+\sqrt{2}y+2=0. \qquad \cdots \qquad \cdots \qquad \cdots \quad \text{(iv)}$$

Now, equation to the tangent to (i) at $(-6, 2\sqrt{2})$ is

$$\frac{x.(-6)}{12}-\frac{y.(2\sqrt{2})}{4}=1,$$

or, $x+\sqrt{2}y+2=0$, the same as (iv). Hence, the tangent at the point is the bisector of the angle between the focal distances of the point.

Ex. 3. *Find the length of the chord of the hyperbola $x^2-4y^2=9$ along the straight line $x+4y+3=0$, and determine the co-ordinates of its middle point.*

At the points of intersection of the hyperbola $x^2-4y^2=9$ ··· (i) with the straight line $x+4y+3=0$ ··· (ii), eliminating x the ordinates are the roots of

$$(4y+3)^2-4y^2=9, \quad \text{or,} \quad y(y+2)=0.$$

$\therefore$ $y=0$, or -2. The corresponding values of x from (ii) are $x=-3$ or 5

Hence, the co-ordinates of the extremities of the chord are $(-3, 0)$ and $(5, -2)$.

Thus, the length of the chord $=\sqrt{(-3-5)^2+(0+2)^2}=2\sqrt{17}$.

Also, the co-ordinates of the middle point of the chord are $[\frac{1}{2}(-3+5), \frac{1}{2}(0-2)]$ *i.e.*, $(1, -1)$.

Ex. 4. *Prove that the portion of the tangent at any point of a hyperbola intercepted between the asymptotes is bisected at the point of contact.*

Let $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ······ (i) be the equation to a hyperbola. Its asymptotes are given by $y=\dfrac{b}{a}x$ ······ (ii) and $y=-\dfrac{b}{a}x$ ······ (iii).

The tangent at any point $P(x', y')$ to the hyperbola (i) is $\dfrac{xx'}{a^2}-\dfrac{yy'}{b^2}=1$ ··· (iv). This meets (ii) at a point Q whose x-co-ordinate [by eliminating y between (ii) and (iv)] is given by

$$\frac{xx'}{a^2}-\frac{y'}{b^2}\cdot\frac{b}{a}x=1, \quad \text{or} \quad x=\frac{a}{\dfrac{x'}{a}-\dfrac{y'}{b}}.$$

Similarly (iv) meets (iii) at R whose x-co-ordinate is given by

$$x=\frac{a}{\dfrac{x'}{a}+\dfrac{y'}{b}}.$$

The x-co-ordinate of the mid-point of QR is

$$\frac{1}{2}\left[\frac{a}{\dfrac{x'}{a}-\dfrac{y'}{b}}+\frac{a}{\dfrac{x'}{a}+\dfrac{y'}{b}}\right]=\frac{x'}{\dfrac{x'^2}{a^2}-\dfrac{y'^2}{b^2}}=x \text{ [by (i)].}$$

Similarly, the y-co-ordinate of the mid-point of QR is y'. Thus, P is the mid-point of QR.

Exan… …l

1. Obtain the equation to the hyperbola whose focus is $(a, 0)$, directrix is the straight line $x = \frac{1}{2}a$, and eccentricity is $\sqrt{2}$.
[*H. S. 1960*]

2. Find the equation to the hyperbola, referred to its axes as axes of co-ordinates,

(i) whose eccentricity is $\sqrt{2}$, and distance between its foci 16.

(ii) whose latus rectum is $10\frac{2}{3}$ and distance between focus and directrix is $3\frac{1}{5}$.

3. In the hyperbo… $4x^2 - 9y^2 = 36$, find the lengths of the axes, the co-ordinates of …e foci, the eccentricity and the length of the latus rectum. [*H. S. 1961*]

[Equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Comparing with the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$\text{major-axis} = 2a = 2.3 = 6$$
$$\text{minor-axis} = 2b = 2.2 = 4.$$

A…ain $b^2 = a^2(e^2 - 1)$, or. $4 = 9(e^2 - 1)$. $\therefore$ $e = \frac{\sqrt{13}}{3}$.

Co-ordinates of foci are $(\pm ae, 0)$, *i.e.*, $(\pm$

Latus-rectum $= \frac{2b^2}{a} = \frac{8}{3}$.]

4. A point moves on the plane of the co-ordinate axes so that the difference of its distances from the points $(\pm 3, 0)$ is always 4. Prove that it traces out a hyperbola whose eccentricity and length of latus rectum you are to determine.

5. By transferring the origin suitably, show that the equation $5x^2 - 4y^2 - 20x - 8y - 4 = 0$ represents a hyperbola, and determine its eccentricity, co-ordinates of its foci, and equations to the directrices.

6. Find the co-ordinates of the foci of the hyperbola $x^2 - y^2 = 9$.

Also find the distance from the origin of the point where the tangent to the above hyperbola at $(5, 4)$ meets the x-axis.
[*H. S, 1960, Compartmental*]

[Equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{9}=1$.

$\therefore\ 9=9(e^2-1)$. $\therefore\ e=\sqrt{2}$.

Co-ordinates of foci $(\pm 3.\sqrt{2}, 0)$.

Equation of tangent at (5, 4) is [as in § 8·4].

$$5x-4y=9.$$

When it meets x-axis, $y=0$ *i.e.*, $5x=9$, $\therefore\ x=\frac{9}{5}$.]

7. Show that the tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at each of the points (i) $(-5, \frac{9}{4})$, (ii) $(8, 3\sqrt{3})$ bisects the angle between the focal distances of the corresponding point.

8. Find the length intercepted on the conjugate axis between the tangents at the two extremities of a latus rectum of the hyperbola $7x^2-9y^2=63$.

9. (i) Find the points on the hyperbola $3x^2-5y^2=15$ at which the tangents are inclined at 60° to the x-axis.

(ii) Find the tangents perpendicular to $x+2y=0$ of the hyperbola $7x^2-4y^2=28$, and find the points of contact.

10. Prove that the locus of the point of intersection of any two perpendicular tangents to a hyperbola is a circle.

11. Find the equation to the normal to the hyperbola $16x^2-25y^2=31$ at the point whose ordinate is -3 and abscissa positive.

12. In the rectangular hyperbola $x^2-y^2=a^2$, show that

(i) the intercept on the x-axis of the normal at any point is double the abscissa of the point.

(ii) the length of the normal at any point intercepted between the axes is bisected at the point.

13. Obtain the length of the chord of the hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$, passing through the origin and making equal angles with the axes. [*H. S. 1960, Compartmental*]

[Equation of line through origin making equal angles with axes is $y=x$.

Solving $y=x$ and $25x^2-9y^2=225$, let (x_1, y_1) and (x_2, y_2) be the points of intersections. Then $(x_1, x_2)=\pm\frac{15}{4}$

$$(y_1, y_2)=\pm\frac{15}{4}.$$

$\therefore$ Length of chord $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\frac{225}{4}+\frac{225}{4}=\frac{15}{\sqrt{2}}$.]

14. Find the equation to the chord of the hyperbola $x^2-2y^2=1$ which is bisected at the point $(-3, -1)$.

[Let the equation of chord be $y'=mx+c$.

Since bisected at $(-3, -1)$,

$$\therefore\quad -1=\frac{\frac{1}{2}}{1.m}(-3) \quad [\text{ as in § 8·8 }]$$

$$\therefore\quad m=\tfrac{3}{2}.$$

Equation of chord becomes $y=\frac{3}{2}x+c$.

Since chord passes through $(-3, -1)$,

$$\therefore\quad -1=\tfrac{3}{2}.(-3)+c. \qquad \therefore\quad c=\tfrac{7}{2}.$$

Hence equation of chord is $y=\frac{3}{2}x+\frac{7}{2}$ or $2y=3x+7$.]

15. Find the length of the chord of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ along the line $3x+2y=12$.

16. Find the equation to the diameter of the hyperbola $\frac{x^2}{4}-\frac{y^2}{5}=1$ bisecting all chords parallel to $x-2y+7=0$.

17. If P be a point on a rectangular hyperbola, prove that

$$SP.S'P=CP^2.$$

[Let $P(x_1, y_1)$ be any point on hyperbola $x^2-y^2=a^2$, and S and S' are two foci of the hyperbola.

$SP=e.PM$ where PM is the length of the perpendicular from P on the directrix

$$x=\frac{a}{e}=e\cdot\frac{x_1-\frac{a}{e}}{\sqrt{1}}=e.\left(x_1-\frac{a}{e}\right)$$

Similarly $S'P = e\left(x_1 + \frac{a}{e}\right)$.

$\therefore \quad SP.S'P = e^2\left(x_1^2 - \frac{a^2}{e^2}\right) = 2\left(x_1^2 - \frac{a^2}{2}\right)$

$= 2x_1^2 - a^2$ [since $e^2 = 2$ for rectangular hyperbola]

$= 2x_1^2 - (x_1^2 - y_1^2)^2$ [since $x_1^2 - y_1^2 = a^2$]

$= (x_1^2 + y_1^2)$

$= CP^2$.]

18. The normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes ; prove that the locus of P is the hyperbola

$$a^2x^2 - b^2y^2 = (a^2 + b)^2.$$

ANSWERS

1. $2x^2 - 2y^2 = a^2$. **2.** (i) $x^2 - y^2 = 32$. (ii) $\frac{x^2}{9} - \frac{y^2}{16} =$

3. 6, 4 ; $(\pm\sqrt{13}, 0)$; $\frac{1}{3}\sqrt{13}$; $2\frac{2}{3}$. **4.** $\frac{3}{2}$; 5.

5. $\frac{3}{2}$; $(5, -1)$ and $(-1, -1)$; $x = 3\frac{1}{3}$ and $x = \frac{2}{3}$. **6.** $(\pm 3\sqrt{2}, 0)$; $1\frac{1}{5}$.

8. 6. **9.** (i) $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{5}{2}, -\frac{\sqrt{3}}{2}\right)$.

(ii) $y = 2x \pm 3$; $(\frac{8}{3}, \frac{7}{3})$ and $(-\frac{8}{3}, -\frac{7}{3})$. **11.** $75x - 64y = 492$.

13. $\frac{15}{2}\sqrt{2}$. **14.** $3x - 2y + 7 = 0$. **15.** $\frac{4}{3}\sqrt{13}$. **16.** $5x - 2y = 0$.

ELECTIVE MATHEMATICS, PAPER II

HIGHER SECONDARY EXAMINATION PAPERS

Board of Secondary Education, W. B.

1962

GROUP A

1. (*a*) Prove that in an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle, together with twice the rectangle contained by one of these sides and the projection of the other side on it.

(*b*) Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

2. (*a*) If two chords of a circle intersect inside the circle, prove that the rectangle contained by the parts of one, is equal to the rectangle contained by the parts of the other.

(*b*) Through any point X on the common chord of two intersecting circles, chords AB and CD are drawn one in each circle. Prove that $AX.XB = CX.XD$.

3. (*a*) Prove that if two triangles are equiangular their corresponding sides are proportional.

(*b*) In the trapezium $ABCD$, AB is parallel to DC, and the diagonals intersect at O. Show that $OA : OC = OB : OD$.

4. (*a*) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) AD is a median of the triangle ABC, and the angles ADB, ADC are bisected by lines which meet AB, AC at E and F respectively. Show that EF is parallel to BC.

5. Costruct a regular hexagon circumscribing a circle of radius 1·5 inches. Measure a side of the hexagon.

[*Statement of construction as well as justification, are to be given.*]

GROUP B

6. (*a*) Find the co-ordinates of the point which divides in a given ratio $m_1 : m_2$ internally, the line joining two given points (x_1, y_1) and (x_2, y_2).

(b) The co-ordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). Find co-ordinates of the point where the medians of the triangle intersect.

7. (a) Find the angle between the straight lines whose equations are $y=m_1v+c_1$ and $y=m_2x+c_2$.

(b) Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the line $5x-2y+7=0$.

8. (a) Find the equation of the circle passing through the origin which makes intercepts 6 and 8 on the positive sides of the axes of x and y respectively.

(b) Prove that the centres of the three circles

$$x^2+y^2-2x+6y=-1$$
$$x^2+y^2+4x-12y=9$$
$$\text{and} \qquad x^2+y^2-16=0$$

lie on a straight line.

9. (a) Find the equation of the parabola, whose focus is at the point $(5, 0)$ and whose directrix is the line $3x-4y+2=0$.

(a) Show that the straight line $y=mx+\frac{a}{m}$ is a tangent to the parabola $y^2=4ax$.

10. (a) Find the equation of the ellipse whose major and minor axes lie along the axes of co-ordinates OX, OY respectively and whose eccentricity is $\frac{1}{\sqrt{2}}$ and latus rectum is 3.

(b) Show that the line $x-y=5$ touches the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$

GROUP C

11. Prove that all straight lines drawn perpendicular to a given straight line at a given point are coplanar.

12. If a right angle rotates about one of its arms, prove that the other arm describes a plane.

13. Find the volume and the lateral surface of a right prism 8 inches long, standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side 6 inches.

14. A right pyramid stands on a rectangular base whose sides are 12 inches and 9 inches; and the length of each of the slant edges is 8·5 inches. Find the height and the volume of the pyramid.

1963

GROUP A

1. (a) If two triangles have their sides proportional, when taken in order, prove that they are equiangular.

(b) Prove that the areas of two similar triangles are proportional to the squares on their circum-radii.

2. (a) If the base of a triangle be divided externally in the ratio of the other two sides, prove that the line joining the vertex to this point of division bisects the vertical angle externally.

(b) Prove that the external bisectors of two angles and the internal bisector of the third angle are concurrent.

3. (a) Show that the acute angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(b) Two circles intersect at A and B, and through P, any point on one of them, straight lines PAC and PBD are drawn to cut the other at C and D. Show that CD is parallel to the tangent at P.

4. Construct, to the scale, an isosceles triangle with each of the equal sides equal to 2 inches, and each base angle double the vertical angle.

Or,

Divide a straight line of length 2 inches into two parts, such that the square on one part may be three times the square on the other.

[*Statement of construction and full neat traces are to be given in any one of the above cases, but no proof.*]

GROUP B

5. (a) Obtain the distance between two points whose rectangular Cartesian co-ordinates are (x_1, y_1) and (x_2, y_2).

(b) Prove that three times the sum of the squares on the side of a the successive angular points of a rectangle.

6. (a) Obtain the perpendicular distance from the point (x_1, y_1) to the straight line $ax+by+c=0$.

(b) Find the orthocentre of the triangle whose angular points are (2, 7), (−6, 1) and (4, −5).

7. (a) Find the equation to the tangent at (x_1, y_1) of the circle

$$x^2+y^2=a^2.$$

(b) Obtain the equation to the circle which passes through the point (0, 4) and touches the x-axis at the point (2, 0).

8. (a) A tangent to the parabola $y^2=12x$ makes an angle 45° to the axis. Find the co-ordinates of its point of contact.

(b) The co-ordinates of the foci of a hyperbola are (5, 0), and (−5, 0) and its eccentricity is $\frac{5}{3}$. Find its equation.

9. (a) Show that the locus of the middle points of a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a straight line passing through its centre.

(b) Find the equation to the normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ at an extremity of a latus rectum.

GROUP C

10. (a) If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane in which they lie.

(b) If $PA=PB=PC$, where P is a point outside the plane of the triangle ABC, and if PO be drawn perpendicular to the plane, prove that O is the circum-centre of the triangle ABC.

(c) If two straight lines are both perpendicular to a plane, show that they are parallel.

(d) If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

11. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised ?

1964

GROUP A

1. Construct a square equal in area to a given rectangle.

Or, Construct a regular hexagon about a given circle,

(Traces of construction only are required in *either* of the two constructions.)

2. Prove that in every triangle, the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing the angle diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.

ABC is an isosceles triangle and AY is drawn to cut the base internally at Y. Show that $AY^2 = AB^2 - BY.YC$.

3. Prove that, if two triangles are equiangular. their corresponding sides are proportional.

Prove that the altitudes of two similar triangles are proportional to the corresponding sides.

4. Prove that if two chords of a circle intersect inside the circle, the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other.

In a triangle ABC, perpendiculars AP and BQ are drawn from A and B to opposite sides and intersect at O. Prove that

$$AO.OP = BO.OQ.$$

GROUP B

5. Find the co-ordinates of the point which divides the straight line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio of $m : n$.

Write down the co-ordinates of the middle point of the straight line joining the points $(7, -4)$ and $(-5, 6)$.

6. Find the equation of the straight line passing through the intersection of the straight lines $2x-7y+11=0$ and $x+3y-8=0$ if it

(a) passes through the origin ;

(b) is perpendicular to the straight line $2x-5y+6=0$;

(c) makes equal intercepts on the two axes.

7. Prove that the straight line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y-12=0$ and find the co-ordinates of the point of contact.

8. Find the focus, vertex and directrix of the parabola

$$(y+3)^2=2(x+2).$$

9. What do you understand by the term 'eccentricity' as applied to a hyperbola ?

Find the equation of the hyperbola whose focus is (2, 3), and directrix, the line $x+2y=1$ and eccentricity $\sqrt{3}$.

GROUP C

10. Give instances from the sides and edges of a cube of :

(a) parallel planes, (b) planes perpendicular to one another, (c) lines parallel to a plane, (d) lines perpendicular to a plane, (e) pairs of skew lines.

Or, Prove that if a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

11. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively. Find the height and the area of the total surface of the prism.

Or, A conical tent is required to accommodate 4 people ; each person must have 20 sq. ft. of space on the ground and 100 cu. ft. of air to breathe. Find the height and radius of the tent. $[\pi=\frac{22}{7}.]$